Continuous linear and quadratic differential systems on the 2-dimensional torus

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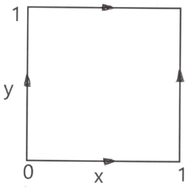
This is a joint work with Ali Bakhshalizadeh

October 10, 2022

1. Linear and quadratic systems on $\mathbb{T}^2$
2. Equilibrium points of the continuous QS
3. Limit cycles

The 2-dimensional torus

We identify the 2-dimensional torus $\mathbb{T}^2$ with $(\mathbb{R}/\mathbb{Z})^2$, i.e.



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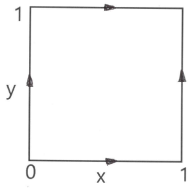

$(x, 0) = (x, 1)$ for all $x \in [0, 1]$, and $(0, y) = (1, y)$ for all $x \in [0, 1]$.

A continuous linear differential system on the torus $\mathbb{T}^2$ is of the form

$$\dot{x} = a + bx + cy, \qquad \dot{y} = A + Bx + Cy,$$

satisfyin

$$\dot{x}|_{x=0} - \dot{x}|_{x=1} = -b = 0, \qquad \dot{y}|_{x=0} - \dot{y}|_{x=1} = -B = 0,$$
$$\dot{x}|_{y=0} - \dot{x}|_{y=1} = -c = 0, \qquad \dot{y}|_{y=0} - \dot{y}|_{y=1} = -C = 0.$$

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Then the continuous linear differential systems on the torus $\mathbb{T}^2$ are

$$\dot{x} = a, \qquad \dot{y} = A,$$

In fact these differential systems on the torus are analytic.

These systems depend on 2 parameters, while the linear differential systems on the plane $\mathbb{R}^2$ depend on 6 parameters.

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So the study of the continuous linear differential systems on the torus $\mathbb{T}^2$ is easier than the study of the linear differential systems on the plane $\mathbb{R}^2$.

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A. DENJOY, Sur les courbes définies par les équations différentielles a la surface de la tore, J. Mathématiques Pures et Appliquées, ser. 9, **11** (1932), 333–375.

C. L. SIEGEL, On differential equations on the torus, Annals of Mathematics **46** (1945), 423–428.

A continuous quadratic differential system on the torus $\mathbb{T}^2$ is of the form

$$\begin{aligned}\dot{x} &= a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2,\\ \dot{y} &= b_0 + b_1 x + b_2 y + b_3 x^2 + b_4 xy + b_5 y^2,\end{aligned}$$

satisfying

$$\begin{aligned}\dot{x}|_{x=0} - \dot{x}|_{x=1} &= -a_1 - a_3 - a_4 y = 0, & \dot{y}|_{x=0} - \dot{y}|_{x=1} &= -b_1 - b_3 \\ \dot{x}|_{y=0} - \dot{x}|_{y=1} &= -a_2 - a_5 - a_4 x = 0, & \dot{y}|_{y=0} - \dot{y}|_{y=1} &= -b_2 - b_5\end{aligned}$$

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Then the continuous quadratic differential systems on the torus $\mathbb{T}^2$ are

$$\dot{x} = a_0 + a_3 x(x-1) + a_5 y(y-1),$$
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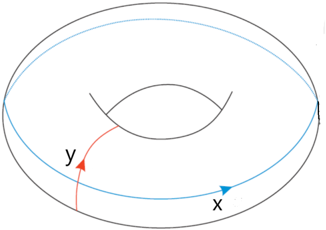
In summary on the red and blue circles in the torus the quadratic system is only continuous in the rest it is analytic.



Renaming the parameters the continuous quadratic differential systems on the torus $\mathbb{T}^2$ are

$$\begin{aligned}\dot{x} &= a + bx(x-1) + cy(y-1),\\ \dot{y} &= A + Bx(x-1) + Cy(y-1).\end{aligned}$$

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So the study of the continuous quadratic differential systems on the torus $\mathbb{T}^2$ is easier than the study of the quadratic differential systems on the plane $\mathbb{R}^2$.

If we want that the continuous quadratic differential systems on the torus $\mathbb{T}^2$

$$\begin{aligned} \dot{x} &= a + bx(x-1) + cy(y-1) = P(x,y), \\ \dot{y} &= A + Bx(x-1) + Cy(y-1) = Q(x,y). \end{aligned}$$

be additionally C^1 we must impose that the first derivatives of the polynomials P and Q coincide at the points $(x,0)$ and $(x,1)$ for all $x \in [0,1]$, and at the points $(0,y)$ and $(1,y)$ for all $y \in [0,1]$.

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be additionally C^1 we must impose that the first derivatives of the polynomials P and Q coincide at the points $(x,0)$ and $(x,1)$ for all $x \in [0,1]$, and at the points $(0,y)$ and $(1,y)$ for all $y \in [0,1]$.

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Again this system is analytic.

Recall that if a QS in the torus $\mathbb{T}^2$ has no equilibria, then the whole torus is filled with either periodic orbits, or dense orbits.

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We do not consider QS in the torus $\mathbb{T}^2$ with infinitely many equilibria.

Now we shall classify the local phase portraits of the equilibria of all the continuous quadratic differential systems of the torus $\mathbb{T}^2$.

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Recall that these systems depend on 6 parameters, while the quadratic differential systems in the plane $\mathbb{R}^2$ depend of 12 parameters.

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The whole classification of the local phase portraits of the equilibria of the quadratic differential systems in the plane $\mathbb{R}^2$ needs a lot of work, thus the 600 pages of the next book were dedicated to a such classification.

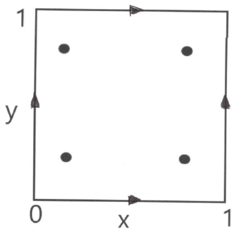
J.C. ARTÉS, J. LLIBRE, D. SCHLOMIUK, N. VULPE, Geometric Configurations of Singularities of Planar Polynomial Differential Systems. A Global Classification in the Quadratic Case, Birkhäuser, 2021.

Assume that $Bc - bC \neq 0$ and that

$$(aC - Ac)(Ab - aB)\left(1 + 4\frac{aC - Ac}{Bc - bC}\right)\left(1 + 4\frac{Ab - aB}{Bc - bC}\right) \neq 0.$$

Then the QS have the following 4 equilibria

$$\left(\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4\frac{aC - Ac}{Bc - bC}}, \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4\frac{Ab - aB}{Bc - bC}}\right),$$



BERLINSKII THEOREM. Assume that a quadratic system

$$\dot{x} = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2,$$
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in the plane $\mathbb{R}^2$ has four equilibria at the vertices of a convex quadrilateral. Then two opposite equilibria are saddles (index -1) and the other two are antisaddles (index 1).

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A. N. BERLINSKII, On the behavior of the integral curves of a differential equation, Izv. Vyssh. Uchebn. Zaved. Mat. **2** (1960), 3–18.

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in the torus $\mathbb{T}^2$ has four equilibria. Then they are localized at the vertices of a rectangle with center at the point $(1/2, 1/2)$. Two opposite equilibria are saddles (index -1) and the other two are antisaddles (index 1). The two antisaddles are both either nodes, or foci, or centers, these three possibilities are realizable.

The four equilibria are

$$\left(\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4\frac{aC - Ac}{Bc - bC}}, \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4\frac{Ab - aB}{Bc - bC}}\right),$$

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We define

$$K=\frac{1}{2}\sqrt{1+4\frac{aC-Ac}{Bc-bC}},\qquad L=\frac{1}{2}\sqrt{1+4\frac{Ab-aB}{Bc-bC}}.$$

If $K>0$, $L>0$ and $(aC-Ac)(Ab-aB)\neq 0$, then the 4 equilibria write

$$\left(\frac{1}{2}\pm K,\frac{1}{2}\pm L\right).$$

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If $(bC - Bc)KL > 0$, $4(Bc - bC)KL + (bK + CL)^2 < 0$, $bK + CL = 0$ and $b^3c - BC^3 = 0$ the equilibrium point is a center.

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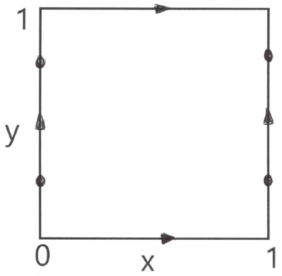

2) If $bC - Bc \neq 0$, $Ab - aB = 0$ and $(aC - Ac)K \neq 0$, then the two equilibria are $(1/2 + K, 0) = (1/2 + K, 1)$ and $(1/2 - K, 0) = (1/2 - K, 1)$.

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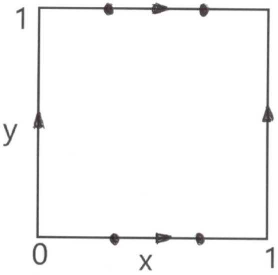

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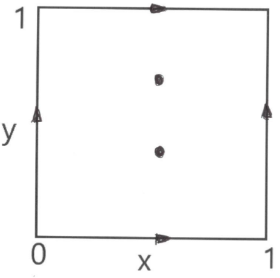

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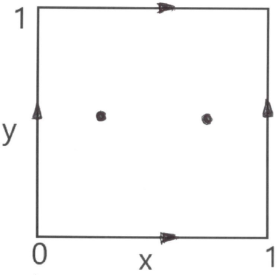

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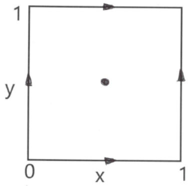

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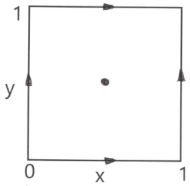

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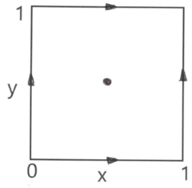

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If we translate this equilibrium to the origin of coordinates, the quadratic system becomes the homogeneous quadratic system $\dot{x} = bx^2 + cy^2$, $\dot{y} = Bx^2 + Cy^2$.

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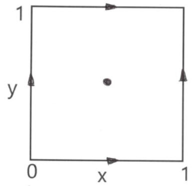

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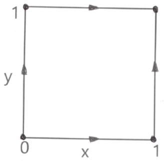
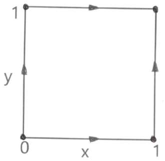
If we translate this equilibrium to the origin of coordinates, the quadratic system becomes the homogeneous quadratic system $\dot{x} = bx^2 + cy^2$, $\dot{y} = Bx^2 + Cy^2$. And all the homogeneous quadratic systems have been classified.



2) If $bC - Bc \neq 0$ and $aC - Ac = Ab - aB = 0$ then the QS has the equilibrium $(0, 0) = (1, 0) = (0, 1) = (1, 1)$.



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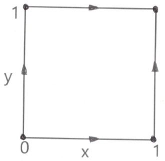

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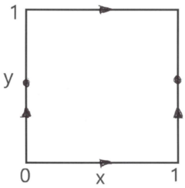
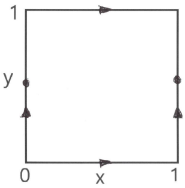


3) If $bC - Bc \neq 0$, $Ab - aB \neq 0$ and $aC - Ac = L = 0$, then the system has the equilibrium point $(0, 1/2) = (1, 1/2)$.



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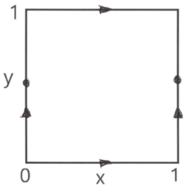

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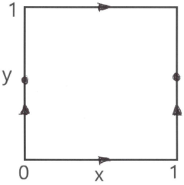

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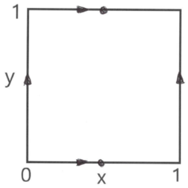

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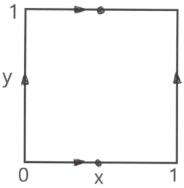

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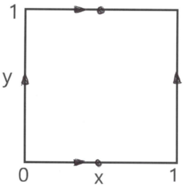

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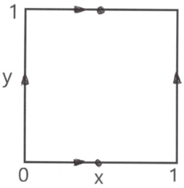

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$bC - Bc = 0$ and either $A = B = C = 0$, or $a = b = c = 0$, or $aB - Ab = 0$, or $aC - Ac = 0$.

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1) There exists a unique equilibrium point in the interior of the region homeomorphic to a disc limited by a periodic orbit. If the periodic orbit is a limit cycle this equilibrium is a focus, and if the periodic orbit is not a limit cycle this equilibrium is a center.

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For a proof of all these properties see the paper:

W.A. Coppel, A Survey of Quadratic Systems, J. Differential Equations **2** (1966), 293–304.

THEOREM. (a) For the continuous QS on the 2-dimensional torus from a Hopf bifurcation at most bifurcates one limit cycle.

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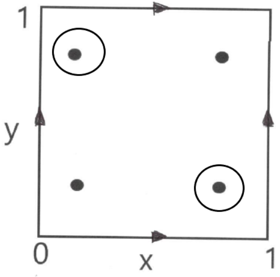
The next configuration of contractible limit cycles to a point is the unique that the continuous QS on the 2-dimensional torus can exhibit.



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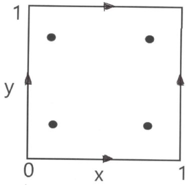
A. Zegeling and R.E. Kooij, The Distribution of limit cycles in quadratic systems with four finite singularities, J. Differential Equations **151** (1999), 373–385.

For the differential system

$$\dot{x} = bx(x-1), \qquad \dot{y} = A + Bx(x-1) + Cy(y-1), \quad \text{with } Ab \neq 0,$$

on the 2-dimensional torus has the circle $x = 0$, or equivalently the circle $x = 1$ as a non-contractible limit cycle.



For the differential system

$$\dot{x} = a + bx(x-1) + cy(y-1), \qquad \dot{y} = Cy(y-1), \quad \text{with } aC \neq 0,$$

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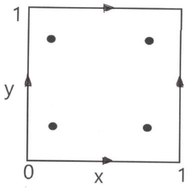

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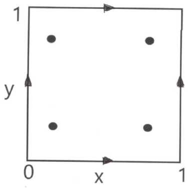

We conjecture that these configurations are all the configurations of the limit cycles for the continuous quadratic differential systems on the torus $\mathbb{T}^2$

The end

THANK YOU VERY MUCH FOR YOUR ATTENTION