Asymptotic recurrence quantification analysis

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Outline

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Recurrence

Recurrence

- one of the fundamental properties of dynamical systems
- introduced by Henri Poincaré in 1890

Poincaré Recurrence Theorem

- Neglecting some exceptional trajectories, the occurrence of which is infinitely improbable, it can be shown, that the system recurs infinitely many times as close as one wishes to its initial state.
- If $(X, \mathcal{B}, \mu, f)$ is a measure-theoretical dynamical system, then for any measurable set A and for μ-a.e. $x \in A$ it holds that

$$f^n(x) \in A \qquad \text{for infinitely many } n \in \mathbb{N}$$

Visualization of recurrence

Recurrence plots (RP)

- introduced by Eckmann, Kamphorst, Ruelle (1987)

Construction:

- fix a DS (X, f), a point $x \in X$ and its trajectory

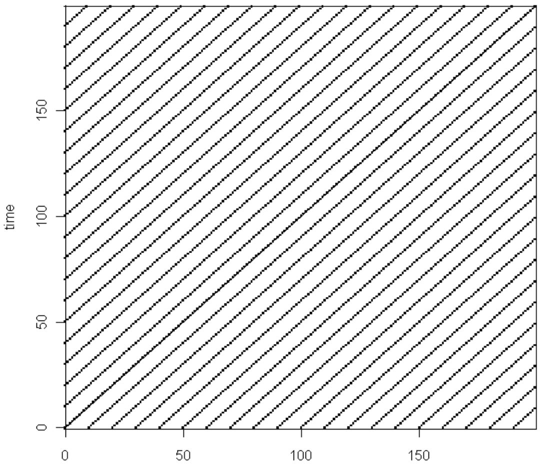
$$x_0 = x,\ x_1 = f(x_0),\ x_2 = f(x_1),\ \ldots,$$

- calculate the $n \times n$ recurrence matrix $RM_n = (R_{ij})_{ij<n}$

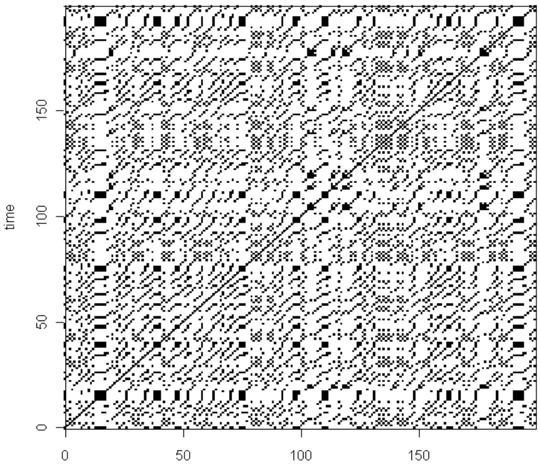
$$R_{ij} = \begin{cases} 1 & \text{if } x_i \approx x_j \\ 0 & \text{if } x_i \not\approx x_j \end{cases} \qquad x_i \approx x_j \iff d(x_i, x_j) \le \varepsilon$$

- recurrence plot: the "black-and-white image" of RM_n
  - black dot at the point (i, j) iff $R_{ij} = 1$ (recurrence)

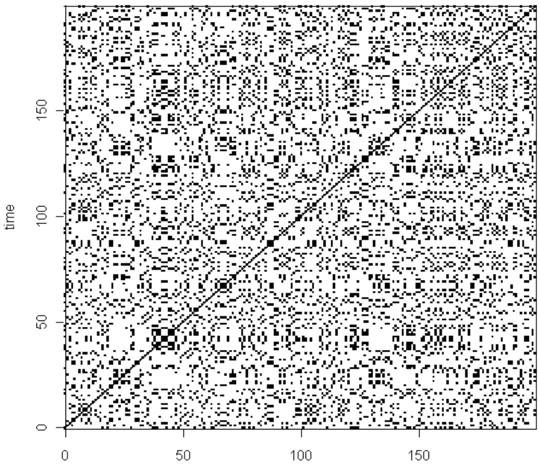
RP of a periodic trajectory (period 10)



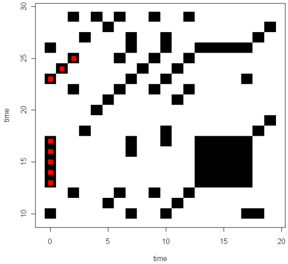
RP of the full logistic map $(f(x) = 4x(1-x),\ x = 0.1,\ \varepsilon = 0.1)$



RP of an RND generator



Patterns in RPs



Diagonal segments (segments parallel to the main diagonal)

- recurrence of a part of the trajectory

Vertical segments

- trajectory "trapped" near fixed points

Outline

Recurrence quantification analysis (RQA)

Recurrence quantification analysis (RQA)

- quantification of structures of RPs
- mainly based on
  - diagonal segments
  - vertical segments

Introduced by

- Zbilut and Webber in 1992

RR: Recurrence rate

RR^k: k-recurrence rate

- density of recurrences in diagonal segments of length $\geq k$

$$RR^k = RR^k_{xn}(\varepsilon) = \frac{2}{n(n-1)} \sum_{l \geq k} l \cdot n_l$$

- n_l: the number of diagonal segments of length l in the $n \times n$ recurrence plot

RR: Recurrence rate

RR^k: k-recurrence rate

- density of recurrences in diagonal segments of length $\geq k$

$$RR^k = RR^k_{xn}(\varepsilon) = \frac{2}{n(n-1)} \sum_{l \geq k} l \cdot n_l$$

  - n_l: the number of diagonal segments of length l in the $n \times n$ recurrence plot

Example

- period 10: $RR^1 = 9.9\%$
- full logistic: $RR^1 = 10.8\%$
- RND: $RR^1 = 9.7\%$

RR: Recurrence rate

Special case: RR^1 = correlation sum

- Grassberger, Procaccia (1983)
- for $n \to \infty$: probability that x returns to its ε-neighborhood

Theorem (Pesin, Tempelman (1995))

If μ is an ergodic measure, then for μ-a.e. $x \in X$ recurrence rates converge (uniformly in ε) to the correlation integral

$$RR^1_{xn}(\varepsilon) \longrightarrow \int_X \mu B(y,\varepsilon)\, d\mu(y) \qquad \text{for } n \to \infty$$

DET: Determinism

DET^k: k-determinism

- the ratio of recurrences in "long" diagonal segments to all recurrences

$$DET^k = DET^k_{xn}(\varepsilon) = \frac{RR^k}{RR^1} = \frac{\sum_{l \geq k} l \cdot n_l}{\sum_{l \geq 1} l \cdot n_l}$$

  - n_l: the number of diagonal segments of length l in the $n \times n$ recurrence plot

Interpretation

- How well one can predict k members of the trajectory based on an observed recurrence?

DET: Determinism

DET^k: k-determinism

- the ratio of recurrences in "long" diagonal segments to all recurrences

$$DET^k = DET^k_{xn}(\varepsilon) = \frac{RR^k}{RR^1} = \frac{\sum_{l \geq k} l \cdot n_l}{\sum_{l \geq 1} l \cdot n_l}$$

Example

- period 10: $DET^5 = 100.0\%$
- full logistic: $DET^5 = 20.2\%$
- RND: $DET^5 = 0.1\%$

Other RQA measures

RQA measures based on diagonal segments

- L_{max}: maximal diagonal segment length
- L_{avg}: average diagonal segment length
- DIV: divergence ($1/L_{max}$)
- $ENTR$: (Shannon) entropy of diagonal segment lengths
- $TREND$: measure of non-stationarity
- $RATIO$: ratio of DET and RR

RQA measures based on vertical segments

- LAM: laminarity
- TT: (average) trapping time
- V_{max}: maximal vertical segment length

Applications of RQA

Nonlinear time series analysis

- linearity and nonlinearity
- determinism, (low-dimensional) chaos and randomness
- noise level, prediction time, ...

Applications of RQA

- life and earth sciences
- chemistry and physics
- finance and economics
- ...

Survey:

- Marwan, Romano, Thiel, Kurths:
  Recurrence plots for the analysis of complex systems
  Physics Reports 438 (2007), 237 – 329

Outline

Asymptotic determinism

Asymptotic RQA measures derived from $DET^k_{xn}(\varepsilon)$

- asymptotic k-determinism: $n \to \infty$
  - based on the whole trajectory
- asymptotic determinism: $k \to \infty$
  - infinite prediction horizon

Asymptotic determinism

Definition (Asymptotic determinism)

For every $\varepsilon > 0$ and $k \in \mathbb{N}$ we define the upper, lower asymptotic k-determinisms by

$$\overline{DET}_x^k(\varepsilon) = \limsup_{n\to\infty} DET_{xn}^k(\varepsilon), \quad \underline{DET}_x^k(\varepsilon) = \liminf_{n\to\infty} DET_{xn}^k(\varepsilon).$$

and upper, lower asymptotic determinisms by

$$\overline{DET}_x(\varepsilon) = \limsup_{k\to\infty} \overline{DET}_x^k(\varepsilon), \quad \underline{DET}_x(\varepsilon) = \liminf_{k\to\infty} \underline{DET}_x^k(\varepsilon).$$

If the corresponding limits exist, we denote them simply by $DET_x^k(\varepsilon)$ and $DET_x(\varepsilon)$.

Asymptotic determinism

Basic questions about asymptotic determinism

- Is the determinism positive or even equal to one?
  - infinitely "predictable" trajectories
- If the determinism is zero, how fast the convergence to zero is?
  - to estimate the maximal "prediction time"

Asymptotic determinism

Proposition

If X is a compact metric space, then for every $\varepsilon > 0$ there is $\eta > 0$:

$$\underline{DET}_x^k(\varepsilon) \geq \eta^k \qquad \textit{for every } x \in X, k \geq 1$$

Asymptotic determinism

Proposition

If X is a compact metric space, then for every $\varepsilon > 0$ there is $\eta > 0$:

$$\underline{DET}_x^k(\varepsilon) \geq \eta^k \qquad \textit{for every } x \in X, k \geq 1$$

Definition

For given x and ε we say that the determinism goes to zero exponentially fast provided there is $\lambda \in (0, 1)$:

$$\overline{DET}_x^k(\varepsilon) \leq \lambda^k \qquad \text{for every } k$$

Outline

Asymptotic RQA and interval dynamics

Setting:

- $X = [0, 1]$ unit interval
- $f : [0, 1] \to [0, 1]$ continuous interval map

Main results

- characterization of Li-Yorke chaotic maps
- characterization of positive entropy maps

Asymptotic RQA and interval dynamics

Setting:

- $X = [0,1]$ unit interval
- $f : [0,1] \to [0,1]$ continuous interval map

Main results

- characterization of Li-Yorke chaotic maps
- characterization of positive entropy maps

Recall that f is Li-Yorke chaotic iff
$\exists$ uncountable set S such that for every $x \neq y$ from S:

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0 \qquad \limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$$

Zero entropy case — finite ω-limit sets

Omega-limit set $\omega_f(x)$

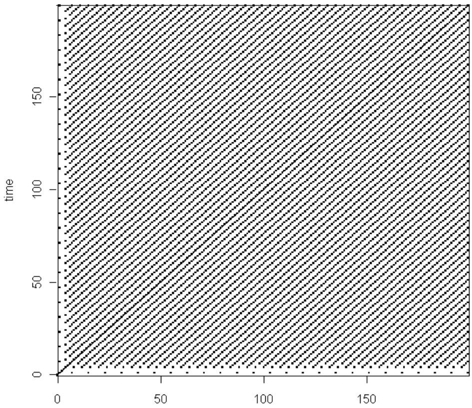
- the set of all limit points of the trajectory $(f^n(x))_{n\geq 0}$ of x

If $\omega_f(x)$ is finite then

- $f|_{\omega_f(x)}$ is a periodic orbit

Zero entropy case — finite ω-limit sets

Example (Logistic map: $f(x) = 3.55x(1-x)$, $x = 0.1$, $\varepsilon = 0.1$)



Zero entropy case — finite ω-limit sets

Lemma

If $\omega_f(x)$ is finite, then $\exists\, \varepsilon_0 > 0$:

$$DET_x(\varepsilon) = 1 \qquad \text{for every } \varepsilon \in (0, \varepsilon_0)$$

Zero entropy case — finite ω-limit sets

Lemma

If $\omega_f(x)$ is finite, then $\exists\, \varepsilon_0 > 0$:

$$DET_x(\varepsilon) = 1 \qquad \text{for every } \varepsilon \in (0, \varepsilon_0)$$

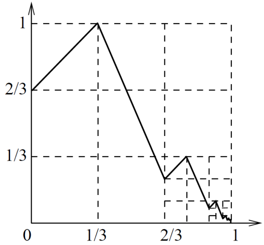
Corollary

If $f : I \to I$ is strongly non-chaotic (that is, f has only finite ω-limit sets), then:

$$DET_x(\varepsilon) = 1 \qquad \text{for every } x \in I \text{ and small } \varepsilon > 0$$

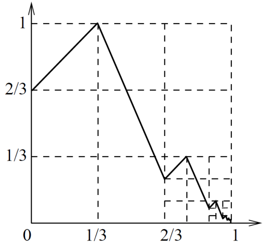
Zero entropy case — infinite ω-limit sets

Example



Zero entropy case — infinite ω-limit sets

Example

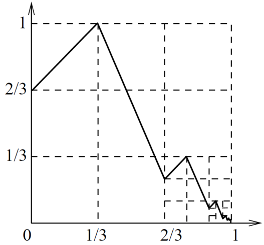


Omega-limit sets

- (unique) 2^p-periodic orbit for every $p \geq 0$
  - $DET = 1$

Zero entropy case — infinite ω-limit sets

Example



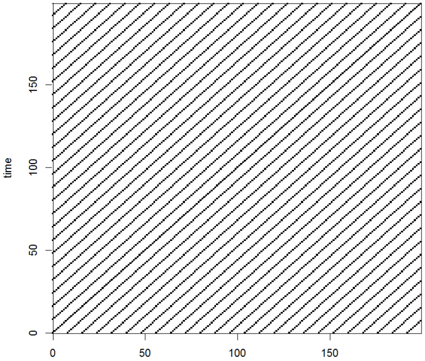
Omega-limit sets

- C: Cantor ternary set
  - $f|_C$ is conjugate to the dyadic adding machine τ
    - τ is an isometry, hence it has $DET = 1$

Zero entropy case — infinite ω-limit sets

Example

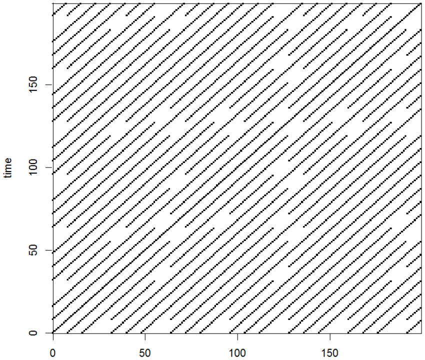
Recurrence plot of $x = 0$, $\varepsilon = \frac{1}{9}$:



Zero entropy case — infinite ω-limit sets

Example

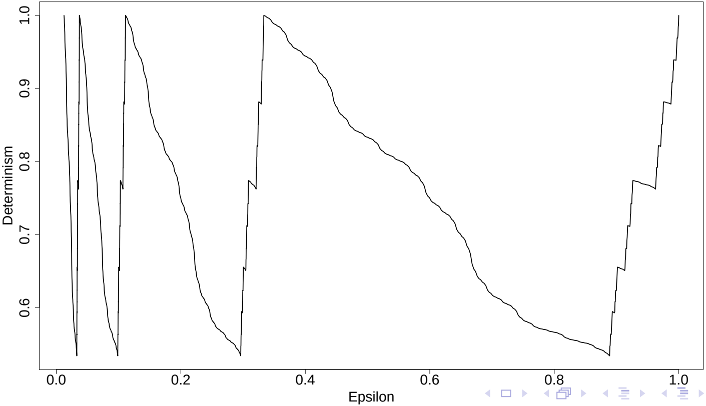
Recurrence plot of $x = 0$, $\varepsilon = \frac{1}{9} - \frac{1}{81}$:



Zero entropy case — infinite ω-limit sets

Example

Dependance of determinism on ε (for $x = 0$)



Zero entropy case — infinite ω-limit sets

Example

Properties of determinism of x with $\omega_f(x) = C$:

- $DET_x(\varepsilon/3) = DET_x(\varepsilon)$ for every $\varepsilon \leq 1$
- $5/8 \leq DET_x(\varepsilon) \leq 1$ for every $\varepsilon > 0$
  - maxima at $\varepsilon = \frac{1}{3^k}$ $(k \geq 0)$
  - minima at $\varepsilon = \frac{1}{3^k} - \frac{1}{3^{k+2}}$ $(k \geq 0)$
- $DET_x(\cdot)$ is
  - strictly decreasing on $[1/3, 8/9]$
  - "Cantor stairs"-like on $[8/9, 1]$

Zero entropy case — not Li-Yorke chaotic maps

Lemma

Let f have zero entropy. If $\omega_f(x)$ contains no two f-non separable points, then

$$\underline{DET}_x(\varepsilon) > 0 \qquad \text{for every } \varepsilon > 0$$

- points y, z are f-separable if
  $\exists$ disjoint periodic intervals $J \ni y$, $K \ni z$
- otherwise: y, z are f-non separable

Zero entropy case — not Li-Yorke chaotic maps

Lemma

Let f have zero entropy. If $\omega_f(x)$ contains no two f-non separable points, then

$$\underline{DET}_x(\varepsilon) > 0 \qquad \text{for every } \varepsilon > 0$$

Proof.

By [Smítal, 1986]

- the trajectory of x is approximable by periodic orbits

□

Zero entropy case — not Li-Yorke chaotic maps

Lemma

Let f have zero entropy. If $\omega_f(x)$ contains no two f-non separable points, then

$$\underline{DET}_x(\varepsilon) > 0 \qquad \text{for every } \varepsilon > 0$$

Corollary

If f is not Li-Yorke chaotic then

$$\underline{DET}_x(\varepsilon) > 0 \qquad \text{for every } x \in I, \varepsilon > 0$$

Zero entropy case — Li-Yorke chaotic maps

Lemma

Let f have zero entropy. If $\omega_f(x)$ contains two f-non separable points y, z, then

$$DET_x(\varepsilon) = 0 \qquad \text{for every } 0 < \varepsilon < |y - z|$$

Zero entropy case — Li-Yorke chaotic maps

Lemma

Let f have zero entropy. If $\omega_f(x)$ contains two f-non separable points y, z, then

$$DET_x(\varepsilon) = 0 \qquad \text{for every } 0 < \varepsilon < |y - z|$$

Proposition

If f has zero entropy and is Li-Yorke chaotic then

$$DET_x(\varepsilon) = 0 \qquad \text{for some } x \in I \text{ and every small } \varepsilon > 0$$

Moreover, for no point x the determinism goes to zero exponentially fast.

Positive entropy case

Lemma

If B is a basic ω-limit set then
$\exists$ (uncountably many) $x \in B$:

$$\overline{DET}_x^k(\varepsilon) \to 0 \quad \textit{exponentially fast for } k \to \infty$$

- B is a basic ω-limit set if
  - it is an infinite ω-limit set
  - contains a periodic point
  - it is maximal (with respect to inclusion)

Positive entropy case

Lemma

If B is a basic ω-limit set then
$\exists$ (uncountably many) $x \in B$:

$$\overline{DET}_x^k(\varepsilon) \to 0 \quad \textit{exponentially fast for } k \to \infty$$

Ingredients of the proof.

- Blokh's theorem about dynamics of $f|_B$
- existence of horseshoes
- the theorem of Pesin-Tempelman

□

Positive entropy case

Proposition

$f : I \to I$ has positive entropy if and only if
$\exists$ (uncountably many) $x \in I$:

$$\overline{DET}_x^k(\varepsilon) \to 0 \quad \textit{exponentially fast for } k \to \infty$$

Summary

Theorem

Let $f : I \to I$ be continuous. Then:

- *f is not Li-Yorke chaotic iff*

$$\underline{DET}_x(\varepsilon) > 0 \qquad \text{for every } x \text{ and small } \varepsilon > 0$$

Summary

Theorem

Let $f : I \to I$ be continuous. Then:

- *f is Li-Yorke chaotic with zero entropy iff*
  $\exists$ (uncountably many) $x \in X$:

  $$DET_x(\varepsilon) = 0 \qquad \text{for every small } \varepsilon > 0$$

  and for no point x the determinism goes to zero exponentially fast

Summary

Theorem

Let $f : I \to I$ be continuous. Then:

- f *has* *positive entropy* *iff*
  $\exists$ *(uncountably many)* $x \in X$*:*

$$\overline{DET}_x^k(\varepsilon) \to 0 \quad \textit{exponentially fast for } k \to \infty$$

Summary

Strongly non-chaotic maps

- all trajectories are perfectly infinitely predictable

Not Li-Yorke chaotic maps

- all trajectories are infinitely predictable with positive accuracy

Li-Yorke chaotic zero entropy maps

- some trajectories are not infinitely predictable
- all trajectories are predictable with long prediction horizon

Positive entropy maps

- some trajectories are predictable only with short prediction horizon

Thanks for your attention!