Iterated Function Systems on the circle

Pablo G. Barrientos and Artem Raibekas

Universidad de Oviedo (Spain)
Universidade Federal Fluminense (Brasil)

ICDEA: 27 July 2012

A semigroup with identity generated (w.r.t. the composition) by a family of diffeomorphisms $\Phi = \{\phi_1, \ldots, \phi_k\}$ on S^1,

$$\mathrm{IFS}(\Phi) \stackrel{\text{def}}{=} \{h : S^1 \to S^1 \colon\ h = \phi_{i_n} \circ \cdots \circ \phi_{i_1},\ \ i_j \in \{1, \ldots, k\}\} \cup \{\mathrm{id}\}$$

is called iterated function system or shortly IFS.

A semigroup with identity generated (w.r.t. the composition) by a family of diffeomorphisms $\Phi = \{\phi_1, \dots, \phi_k\}$ on S^1,

$$\mathsf{IFS}(\Phi) \stackrel{\text{def}}{=} \{h : S^1 \to S^1\colon\ h = \phi_{i_n} \circ \cdots \circ \phi_{i_1},\ \ i_j \in \{1, \dots, k\}\} \cup \{\mathrm{id}\}$$

is called iterated function system or shortly IFS.

For each $x \in S^1$, we define the orbit of x for $\mathsf{IFS}(\Phi)$ as

$$\mathrm{Orb}_\Phi(x) \stackrel{\text{def}}{=} \{h(x)\colon\ h \in \mathsf{IFS}(\Phi)\} \subset S^1$$

and the set of periodic points of $\mathsf{IFS}(\Phi)$ as

$$\mathrm{Per}(\mathsf{IFS}(\Phi)) \stackrel{\text{def}}{=} \{x \in S^1\colon\ \ h(x) = x \text{ for some } h \in \mathrm{IFS}(\Phi),\ \ h \neq \mathrm{id}\}.$$

Let $\Lambda \subset S^1$. We say that Λ is

- invariant for IFS(Φ) if $\mathrm{Orb}_\Phi(x) \subset \Lambda$ for all $x \in \Lambda$,
- minimal for IFS(Φ) if

$$\Lambda \subset \overline{\mathrm{Orb}_\Phi(x)} \quad \text{for all } x \in \Lambda.$$

Let $\Lambda \subset S^1$. We say that Λ is

- <u>invariant</u> for IFS(Φ) if $\mathrm{Orb}_\Phi(x) \subset \Lambda$ for all $x \in \Lambda$,
- <u>minimal</u> for IFS(Φ) if

$$\Lambda \subset \overline{\mathrm{Orb}_\Phi(x)} \quad \text{for all } x \in \Lambda.$$

In order to define robust properties under perturbations we introduce the following concept of proximity into the set of IFSs. We say that

$$\text{IFS}(\psi_1, \dots, \psi_k) \text{ is } C^r\text{-close to } \text{IFS}(\phi_1, \dots, \phi_k)$$

if ψ_i is C^r-close to ϕ_i for all $i = 1, \dots, k$.

Let $\Lambda \subset S^1$. We say that Λ is

- invariant for IFS(Φ) if $\mathrm{Orb}_\Phi(x) \subset \Lambda$ for all $x \in \Lambda$,
- minimal for IFS(Φ) if

$$\Lambda \subset \overline{\mathrm{Orb}_\Phi(x)} \quad \text{for all } x \in \Lambda.$$

In order to define robust properties under perturbations we introduce the following concept of proximity into the set of IFSs. We say that

$$\mathrm{IFS}(\psi_1, \ldots, \psi_k) \text{ is } C^r\text{-close to } \mathrm{IFS}(\phi_1, \ldots, \phi_k)$$

if ψ_i is C^r-close to ϕ_i for all $i = 1, \ldots, k$. So, we will say that

$$S^1 \text{ is } \underline{C^r\text{-robust minimal}} \text{ for } \mathrm{IFS}(\Phi)$$

if S^1 is minimal for all IFS(Ψ) C^r-close enough to IFS(Φ).

Taking into account the rotation number of a homeomorphism $f : S^1 \to S^1$ we have three possibilities:

- f has a periodic orbit,
- all the orbits (for forward iterates) of f are dense,
- there is a wandering interval for f.

The wandering intervals are the gaps of a unique f-invariant minimal Cantor set $\Lambda \subset S^1$.

Taking into account the rotation number of a homeomorphism $f : S^1 \to S^1$ we have three possibilities:

- $\mathrm{IFS}(f)$ has a finite orbit,
- all the orbits (for forward iterates) of f are dense,
- there is a wandering interval for f.

The wandering intervals are the gaps of a unique f-invariant minimal Cantor set $\Lambda \subset S^1$.

Taking into account the rotation number of a homeomorphism $f : S^1 \to S^1$ we have three possibilities:

- $\mathrm{IFS}(f)$ has a finite orbit,
- S^1 is minimal for $\mathrm{IFS}(f)$,
- there is a wandering interval for f.

The wandering intervals are the gaps of a unique f-invariant minimal Cantor set $\Lambda \subset S^1$.

Taking into account the rotation number of a homeomorphism $f : S^1 \to S^1$ we have three possibilities:

- IFS(f) has a finite orbit,
- S^1 is minimal for IFS(f),
- there exists an invariant minimal Cantor set for IFS(f). In this case it is unique.

Taking into account the rotation number of a homeomorphism $f : S^1 \to S^1$ we have three possibilities:

- $\mathrm{IFS}(f)$ has a finite orbit,
- S^1 is minimal for $\mathrm{IFS}(f)$,
- there exists an invariant minimal Cantor set for $\mathrm{IFS}(f)$. In this case it is unique.

This trichotomy can be extended to actions of groups of homeomorphisms on the circle:

<u>THEOREM (Ghys)</u>: Let $\mathrm{G}(\Phi)$ be a subgroup of $\mathrm{Hom}(S^1)$. Then one (and only one) possibility occurs:

- $\mathrm{G}(\Phi)$ has a finite orbit,
- S^1 is minimal for $\mathrm{G}(\Phi)$,
- there exists an invariant minimal Cantor set for $\mathrm{G}(\Phi)$. In this case it is unique.

THEOREM (DENJOY): There exists $\varepsilon > 0$ such that if $f \in \mathrm{Diff}^2(S^1)$ is ε-close to the identity in the C^2-topology then there are no invariant minimal Cantor sets for IFS(f).

Moreover, the following conditions are equivalent:

1. S^1 is minimal for IFS(f),
2. there are no periodic points for f.

THEOREM (Denjoy): There exists $\varepsilon > 0$ such that if $f \in \mathrm{Diff}^2(S^1)$ is ε-close to the identity in the C^2-topology then there are no invariant minimal Cantor sets for $\mathrm{IFS}(f)$.

Moreover, the following conditions are equivalent:

1. S^1 is minimal for $\mathrm{IFS}(f)$,
2. there are no periodic points for f.

THEOREM (Generalized Duminy): There exists $\varepsilon > 0$ such that if $f_0, f_1 \in \mathrm{Diff}^2(S^1)$ are Morse-Smale ε-close to the identity in the C^2-topology then there are no invariant minimal Cantor sets for all $\mathrm{G}(\Psi)$ C^1-close to $\mathrm{G}(f_0, f_1)$.

Moreover, the following conditions are equivalent[a]:

1. S^1 is C^1-robust minimal for $\mathrm{G}(f_0, f_1)$,
2. $f_1(\mathrm{Per}(f_0)) \neq \mathrm{Per}(f_0)$.

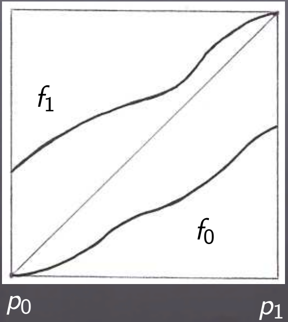
[a] Condition (2) is satisfied if f_0 and f_1 have not periodic points in common.

ss-intervals for IFS(Φ)

DEFINITION: Given $\Phi = \{f_0, f_1\} \subset \mathrm{Diff}^1_+(\mathbb{R})$, an interval $[p_0, p_1] \subset \mathbb{R}$ is called ss-interval for IFS(Φ) if:

- $[p_0, p_1] = f_0([p_0, p_1]) \cup f_1([p_0, p_1])$,
- $(p_0, p_1) \cap \mathrm{Fix}(f_i) \neq \emptyset$ for $i = 1, 2$, and $p_j \notin \mathrm{Fix}(f_i)$ for $i \neq j$,
- p_0 and p_1 are attracting fixed points of f_0 and f_1 resp.

We will denote by K^{ss}_{Φ} a ss-interval $[p_0, p_1]$ for IFS(Φ).



Improved Duminy's Lemma

<u>THEOREM</u>: Let K^{ss}_{Φ} be a ss-interval for IFS(Φ) with $\Phi = \{f_0, f_1\} \subset \mathrm{Diff}^2_+(\mathbb{R})$ such that $f_i|_{K^{ss}_{\Phi}}$ has hyperbolic fixed points. Then, there exists $\varepsilon \geq 0.16$ such that if $f_0|_{K^{ss}_{\Phi}}, f_1|_{K^{ss}_{\Phi}}$ are ε-close to the identity in the C^2-topology, it holds

$$K^{ss}_{\Psi} \subset \overline{\mathrm{Per}(\mathrm{IFS}(\Psi))} \quad \text{and} \quad K^{ss}_{\Psi} = \overline{\mathrm{Orb}_{\Psi}(x)} \quad \text{for all } x \in K^{ss}_{\Psi},$$

for every IFS(Ψ) C^1-close to IFS(Φ).

<u>THEOREM</u>: Consider IFS(Φ) with $\Phi = \{\phi_1, \dots, \phi_k\} \subset \mathrm{Hom}(S^1)$. Then exists a non-empty closed set $\Lambda \subset S^1$ such that

$$\Lambda = \phi_1(\Lambda) \cup \cdots \cup \phi_k(\Lambda) = \overline{\mathrm{Orb}_\Phi(x)} \quad \text{for all } x \in \Lambda.$$

One (and only one) of the following possibilities occurs:

1. Λ is a finite orbit,
2. Λ has non-empty interior,
3. Λ is a Cantor set.

THEOREM: Consider IFS(Φ) with $\Phi = \{\phi_1, \dots, \phi_k\} \subset \mathrm{Hom}(S^1)$. Then exists a non-empty closed set $\Lambda \subset S^1$ such that

$$\Lambda = \phi_1(\Lambda) \cup \cdots \cup \phi_k(\Lambda) = \overline{\mathrm{Orb}_\Phi(x)} \quad \text{for all } x \in \Lambda.$$

One (and only one) of the following possibilities occurs:

1. Λ is a finite orbit,
2. Λ has non-empty interior,
3. Λ is a Cantor set.

Denjoy's Theorem for IFSs

THEOREM: There exists $\varepsilon > 0$ s.t. if $f_0, f_1 \in \mathrm{Diff}^2(S^1)$ are Morse-Smale diff. ε-close to the identity in the C^2-topology with no periodic point in common then, there are no invariant minimal Cantor sets for all IFS(Ψ) C^1-close to IFS(f_0, f_1).

Moreover, denoting by n_i the period of f_i, it is equivalent:

1. S^1 is C^1-robust minimal for IFS($f_0^{n_0}, f_1^{n_1}$),
2. there are no ss-intervals for IFS($f_0^{n_0}, f_1^{n_1}$).

Let $x \in S^1$. The ω-limit of x for $\mathsf{IFS}(\Phi)$ is the set

$$\omega_\Phi(x) \stackrel{\text{def}}{=} \{y \in S^1 : \exists (h_n)_n \subset \mathsf{IFS}(\Phi) \setminus \{\mathrm{id}\} \text{ s.t. } \lim_{n\to\infty} h_n \circ \cdots \circ h_1(x) = y\},$$

while the $\underline{\omega\text{-limit}}$ of $\mathsf{IFS}(\Phi)$ is

$$\omega(\mathsf{IFS}(\Phi)) \stackrel{\text{def}}{=} \mathrm{cl}\big(\{y \in S^1 : \ \exists x \in S^1 \text{ s.t. } y \in \omega_\Phi(x)\}\big),$$

where "cl" denotes the closure of a set. Similarly we define the α-limit of $\mathsf{IFS}(\Phi)$. Finally, the $\underline{\text{limit set}}$ of $\mathsf{IFS}(\Phi)$

$$L(\mathsf{IFS}(\Phi)) = \omega(\mathsf{IFS}(\Phi)) \cup \alpha(\mathsf{IFS}(\Phi)).$$

Let $x \in S^1$. The ω-limit of x for $\mathrm{IFS}(\Phi)$ is the set

$$\omega_\Phi(x) \stackrel{\text{def}}{=} \{y \in S^1 : \exists (h_n)_n \subset \mathrm{IFS}(\Phi) \setminus \{\mathrm{id}\} \text{ s.t. } \lim_{n\to\infty} h_n \circ \cdots \circ h_1(x) = y\},$$

while the <u>ω-limit</u> of $\mathrm{IFS}(\Phi)$ is

$$\omega(\mathrm{IFS}(\Phi)) \stackrel{\text{def}}{=} \mathrm{cl}\big(\{y \in S^1 : \ \exists x \in S^1 \text{ s.t. } y \in \omega_\Phi(x)\}\big),$$

where "cl" denotes the closure of a set. Similarly we define the α-limit of $\mathrm{IFS}(\Phi)$. Finally, the <u>limit set</u> of $\mathrm{IFS}(\Phi)$

$$L(\mathrm{IFS}(\Phi)) = \omega(\mathrm{IFS}(\Phi)) \cup \alpha(\mathrm{IFS}(\Phi)).$$

Let $\Lambda \subset S^1$. We say that Λ is

- <u>transitive</u> for $\mathrm{IFS}(\Phi)$ if there exists a dense orbit in Λ,
- <u>isolated</u> for $\mathrm{IFS}(\Phi)$ if $\Lambda \cap \mathrm{Per}(\mathrm{IFS}(\Phi)) \neq \emptyset$ and there exists an open set D such that

$$\Lambda \subset D \quad \text{and} \quad \overline{\mathrm{Per}(\mathrm{IFS}(\Phi)) \cap D} \subset \Lambda.$$

spectral decomposition for IFs

THEOREM: There exists $\varepsilon > 0$ such that if $f_0, f_1 \in \mathrm{Diff}^2(S^1)$ are Morse-Smale diffeomorphisms of periods n_0 and n_1, respectively, ε-close to the identity in the C^2-topology and with no periodic point in common, then there are finitely many isolated, transitive pairwise disjoint intervals $K_1, \ldots, K_m$ for $\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})$ such that

$$L(\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})) = \bigcup_{i=1}^{m} K_i.$$

Moreover, this decomposition is C^1-robust.

Thanks for your attention