Perturbations on autonomous and non-autonomous systems

Francisco Balibrea

balibrea@um.es
Departamento de Matemáticas
Universidad de Murcia (Spain)

ICDEA, Barcelona. July 2012

Two introductory examples

$$x_{n+1} = ax_n$$

where $a > 0$

$$x_{n+1} = \frac{a + x_n}{x_{n-1}}$$

where also $a > 0$

perturbations

If in both equations we perturb the parameters

$$x_{n+1} = (a + p_n)x_n$$

$$x_{n+1} = \frac{(a + p_n) + x_n}{x_{n-1}}$$

we obtain non-autonomous systems which can be formulated by

$$x_{n+1} = f_n(x_n)$$

Non-autonomous discrete systems (n.a.d.s.)

That is by (X, f_∞) where $f_\infty = (f_n)_{n=0}^{\infty}$ and $f_n \in C(X, X)$ for all n
(X, f_∞) is called a *non-autonoumous discrete system* $(n.a.d.s.)$

(n.a.d.s.)

We use the notation

$$f_i^n = f_{i+(n-1)} \circ f_{i+(n-2)} \circ \dots . f_{i+2} \circ f_{i+1} \circ f_i$$

with $i \geq 0$, $n > 0$ and $f_i^0 =$ Identity on X and
$Tr_{f^\infty}(x_0) = (f_0^n)_{n=0}^\infty = (x_n)_{n=0}^\infty$

(n.a.d.s.)

We are dealing with the *stability or instability in the Lyapunov sense* of such systems

Lyapunov exponents for autonomous systems

They were introduced by Aleksandr Lyapunov in 1892 in his Doctoral Memoir: *The general problem of the stability of motion*

It is a extended practice, especially in experimental and applied dynamics, to associate the idea of orbits having a positive Lyapunov exponent with instability and negative Lyapunov exponent with stability of orbits in dynamical system. Stability and instability of orbits are defined in topological terms while Lyapunov exponents is a numerical characteristic calculated all along the orbit

definition of Lyapunov exponents

Definition

Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1-map. For each point x_0 the *Lyapunov exponent* of x_0, $\lambda(x_0)$ is

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \log(|(f^n)'(x_0)|) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log(|f'(x_j)|)$$

where $x_j = f^j(x_0)$ (if the limit exists).

stability and instability in Lyapunov sense

Definition

The forward trajectory $Tr_f(x_0)$ is said to be Lyapunov stable if for any $\epsilon > 0$ there is $\delta > 0$ such that whenever $|y - x_0| < \delta$ is $|f^n(y) - f^n(x_0)| < \epsilon$ for all $n \geq 0$.

stability and instability in Lyapunov sense

Lyapunov instability is equivalent to sensitivity dependence on initial conditions *(sdic)*

Definition

$Tr_f(x_0)$ exhibits (sdic), if there exists $\epsilon > 0$ such that given any $\delta > 0$ there is y holding $|y - x_0| < \delta$ and $N > 0$ such that

$$|f^n(y) - f^n(x_0)| \geq \epsilon$$

for all $n \geq N$

stability and instability in Lyapunov sense

In the following examples, we consider the trajectories of 0 of two maps and obtain that we can have instability trajectories with negative Lyapunov exponents and stable trajectories with positive Lyapunov exponents

Map f introduced by Demir and Koçak

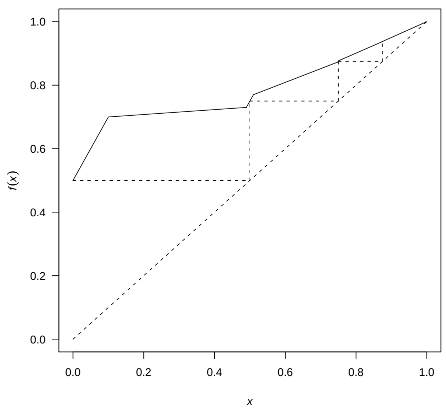


Figure: Map f

Map g introduced by Demir and Koçak

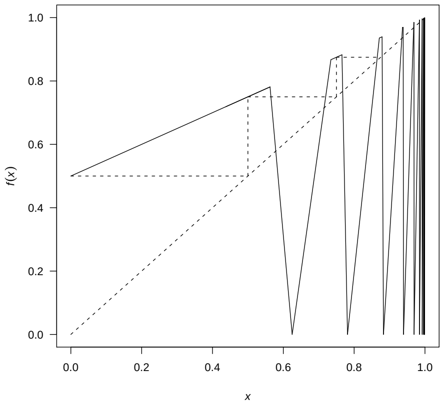


Figure: Map g

The strong Lyapunov exponent is

$$\Phi(x) = lim_{n\to\infty}\frac{1}{n}\Sigma_{j=k}^{k+n-1}\log(|f'(x_j)|)$$

if this limit exists uniformly with respect to $k \geq 0$

Results:

1. Let $f \in C^1(I)$. If the forward trajectory of $x \in I$ has positive strong Lyapunov exponent, then the orbit has (sdic)
2. Let $f \in C^1([0,1))$. If the forward trajectory of $x \in [0,1)$ has negative strong Lyapunov exponent, then the orbit is Lyapunov stable

BC used the notion of Lyapunov exponents for non-autonomous systems on $\mathbb{R}$ and $C^1 - maps$ for the difference equation

$$x_{n+1} = a_n x_n$$

as an immediate extension of the formula to calculate the Lyapunov exponents in the autonomous case (if the limit exists) as

$$\lambda(x) = \lim_{n\to\infty}\frac{1}{n}\log|(f_0^n)'(x)| = \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log|f_j'(x_j)|$$

where $x_j = f_0^j(x)$

Lyapunov exponents for the non-autonomous case

They considered the case when

$$a_n = a + p(n)$$

where

$$p_n = [a + \epsilon(b_n + \beta c_n)$$

holding $a > 1$ but closed to 1, $0 < \beta > 1$ and

$$b_n = \sqrt{2} sinn$$

$$c_n = \sqrt{2} sn[2K(m)(n + \Theta)/\pi; m]$$

with $\epsilon > 0$ and m the modulus of the elipticity of *senam map*

Lyapunov exponents for the non-autonomous case

The Lyapunov exponent has the following values:

1. If $\beta = 0$, then if

$$\log a > \frac{1}{2}(\frac{\epsilon}{a})^2$$

   then the system has for all initial conditions on $(0, \infty)$ constant positive Lyapunov exponents and has (dsic)

2. If $\beta \neq 0$, then for fixed modulus m and in some range of Θ, the system has also constant positive Lyaunov exponents. Also it is proved the system has (dsci)

Stability and instability of orbits in periodic non-autonomous systems

Take a periodic block composed of the maps f and g, $\{f_0, f_1, ..., f_{m-1}\}$ where $p < m$ of them are the map f and the rest g and consider the non-autonomous periodic system of period $m = p + q$ where $f_i = f$ for $i = 0, 1, ..., p-1$ and $f_j = g$ for $j = p, ..., m-1$

If we compute the Lyapunov exponent 0 of such periodic non-autonomous system we have for the f_0^n map
For

$$n = km + 1 \text{ is } \frac{k(p-q)+1}{km+1}\log 2$$
$$n = km + 2 \text{ is } \frac{k(p-q)+2}{km+2}\log 2$$
$$...$$
$$n = km + p \text{ is } \frac{k(p-q)+p}{km+p}\log 2$$
$$...$$
$$(k+1)(p-q)$$

Stability and instability of orbits in periodic non-autonomous systems

When $n \to \infty$, the Lyapunov exponent of the trajectory of 0 is

$$\lambda(0) = \frac{p-q}{m}\log 2$$

Stability and instability of orbits in non-periodic non-autonomous systems

When we choose a non-periodic block of maps f and g the orbit of 0 continues being instable if the map g appears infinite times

Theorem

(BC) Let f_∞ a non-periodic sequence of maps f and g. If the map g appears infinite times, then the trajectory of 0 is Lyapunov instable.

Let $X \subset \mathbb{R}^m$ and d any metric on it. If $(x_n)_{n=0}^{\infty}$ and $(x'_n)_{n=0}^{\infty}$ are two trajectories starting from nearby initial states x_0 and x'_0 and write $\delta x_n = x'_n - x_n$. If f has continuous partial derivatives in every x_i, then, iterating the map, we have the linear approximation ($DF(x)$ denotes the differential of the map $F : \mathbb{R}^n \to \mathbb{R}^n$ at the point x).

$$\delta x_n \simeq Df^n(x_0)\delta x_0 = (\prod_{i=0}^{n-1} Df(x_i)\delta x_0$$

where the (i,j) element of the matrix $Df(x)$ is given by $\frac{\partial f_i}{\partial x_j}$ and where f_i and x_j are the components of f and x in local coordinates on X

Given a matrix A, we denote by A^t the transpose of A. Let the matrix

$$(Df^n(x_0)^t)(Df^n(x_0))$$

where

$$Df^n(x_0) = Df(x_{n-1})(Df(x_{n-2})...(Df(x_1)Df(x_0)$$

have eigenvalues in x_0 given by $\mu_i(n, x_0)$, for $i = 1, 2, ..., m$ such that $\mu_1(n, x_0) \geq \mu_2(n, x_0) \geq ... \geq \mu_m(n, x_0)$. Then the ith local Lyapunov exponent at x_0 is defined by:

$$\lambda_i(x_0) = \lim_{n\to\infty} \frac{1}{2n} \log(|\mu_i(n, x_0)|)$$

if this limit exists. In **[?]** it is possible to state conditions for the existence of such limit. Now we recall the notions of instability and stability in the Lyapunov sense

Markus-Lyapunov Fractal

We consider the logistic equation

$$x_{n+1} = r_n(1 - x_n)$$

and the sequence of blocks $BBBBB....$ where $B = 112112....$ and $112 = r_1 r_1 r_2$

Markus-Lyapunov Fractal

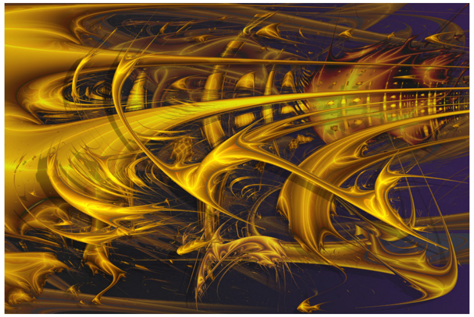
2.jpg 

Figure: Fractal Markus-Lyapunov

Markus-Lyapunov Fractal

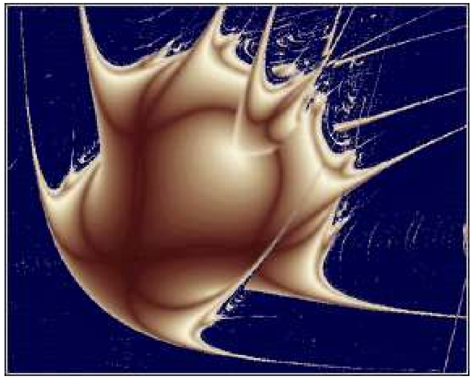
3.jpg 

Figure: Fractal Markus-Lyapunov

We propose two dynamical systems, one defined in $[0, 1]^2$ which has a forward trajectory with a positive Lyapunov exponent but not having sensitive dependence on initial conditions and other defined in $[0, 1)^2$ which has a forward trajectory with a negative Lyapunov exponent but having sensitive dependence on initial conditions. The examples are two dimensional versions of those mentioned in the introduction. The maps we are using are examples of permutation maps.

Example

We are going to obtain a continuous function $F = (f, g)$ in $[0, 1]^2$ such that the forward trajectory of $(0, 0)$ has a positive Lyapunov exponent, but has not has no sensitive dependence on initial conditions.

a) The map $f : [0,1] \to [0,1]$ was introduced in [**?**]

$$f(x) = \begin{cases} 2x - 1 + \dfrac{1}{2^{n+1}} & a_n < x \le b_n,\ x \in \\[2ex] \dfrac{5^{n+2} - 22}{2 \cdot 5^{n+2} - 11}(x - b_n) + 1 + \dfrac{2}{10^{n+1}} - \dfrac{1}{2^{n+1}} & b_n < x \le a_{n+} \\[2ex] 1 & x = 1 \end{cases}$$

with $a_n = 1 - 2^{-n} - 10^{-n-1}$, $b_n = 1 - 2^{-n} + 10^{-n-1}$, $n = 0, 1, 2, \ldots$.

b) Now we define another map $g : [0,1] \to [0,1]$

$$g(x) = \begin{cases} 3x + \dfrac{1}{2} & 0 \le x \le \dfrac{1}{15} \\[2ex] \dfrac{6}{127}x + \dfrac{7}{10} - \dfrac{2}{635} & \dfrac{1}{15} < x \le \dfrac{1}{2} - \dfrac{\ }{1} \\[2ex] 3x + \dfrac{1}{2} - \dfrac{5}{2^{n+1}}(2^n - 1) & a_n < x \le b_n \\[2ex] \dfrac{5^{n+2} - 33}{2 \cdot 5^{n+2} - 11}(x - b_n) + 1 + \dfrac{3}{10^{n+1}} - \dfrac{1}{2^{n+1}} & b_n < x \le a_{n+1} \end{cases}$$

The map $F(x,y) = (f(y), g(x))$ is continuous in $[0,1]^2$, because f y g are continuous in $[0,1]$.
We consider the trajectory of $(0,0)$:

$$\{(0,0), (x_1, y_1), (x_2, y_2), \cdots\}$$

In every points of the trajectory, the map is differentiable (except $(0,0)$)
Since f y g are differentiable maps on right of 0, we define

$$DF(0^+, 0^+) = \begin{pmatrix} 0 & \lim_{y \to 0^+} f(y) \\ \lim_{x \to 0^+} g(x) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix}$$

$$DF(x_1, y_1) = DF^2(0,0) = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$$

$$DF(x_2, y_2) = DF^3(0,0) = \begin{pmatrix} 0 & 12 \\ 18 & 0 \end{pmatrix}$$

and

$$DF^{2n}(0,0) = \begin{pmatrix} 6^n & 0 \\ 0 & 6^n \end{pmatrix} \qquad DF^{2n-1}(0,0) = \begin{pmatrix} 0 & 2 \cdot 6^n \\ 3 \cdot 6^n & 0 \end{pmatrix}$$

for $n = 1, 2, \ldots$.

Now we compute the eigenvalues $(DF^n)^t)(DF^n)$:

- For $n > 1$ we have

$$(DF^{2n-1}(0,0))^t DF^{2n-1}(0,0) = \begin{pmatrix} 3^2 \cdot 6^n & 0 \\ 0 & 2^2 \cdot 6^n \end{pmatrix}$$

  and the maximum value of the eigenvalues of such matrix is

$$\mu(2n-1, (0,0)) = \frac{1}{2n-1}((n+1)\log 3 + n \log 2)$$

- For $n > 1$ we have

$$DF^{2n}(0,0) = \begin{pmatrix} 6^n & 0 \\ 0 & 6^n \end{pmatrix}$$

  whose eigenvalue is

$$\mu(2n, (0,0)) = \frac{1}{2}(\log 3 + \log 2)$$

Therefore,

$$\lambda_1(0,0) = \lim_{n \to \infty} \frac{1}{2n} \log(|\mu_1(n, (0,0))|) = \frac{1}{2} \log 6 > 0$$

it is easy to prove that the forward trajectory of $(0,0)$ has not sensitive dependence on initial conditions

The map is continuous at every $(x,y) \in I^2$.
To see it, let $\epsilon > 0$, we can chose k such that $1/2^k < \epsilon$. As $f'(y) > 0$ and $g'(x) > 0$ on the forward orbit of $(0,0)$, si se considera la distancia del máximo:

$$F^k(0,0) = (1 - \frac{1}{2^m k' m}, 1 - \frac{1}{2^k})$$

then

$$|F^k(x,y) - F^k(0,0)| \leq |(\frac{1}{2^k}, \frac{1}{2^k})| = \frac{1}{2^k} < \epsilon$$

for $n \geq k$ and $0 < x < \bar{\delta}$ it remains to prove that the last inequality holds for $n < k$, but it is made using that F^j is continuous and then, given $\epsilon > 0$ there exists δ_j such that if $0 < |(x,y)| < \delta_j$, $|F^j(x,y) - F^j(0,0)| < \epsilon$ for $j = 1, ..., n-1$. Then if we take

$$\delta = \min\left\{\delta_1, ..., \delta_{n-1}, \bar{\delta}\right\} \text{ and } 0 < x < \delta \Rightarrow |F^k(x,y) - F^k(0,0)| < \epsilon \text{ for all } k$$

Example

We are going to obtain a continuous function $G = (f^2, g)$ in $[0,1)^2$ such that the forward trajectory of $(0,0)$ has a negative Lyapunov exponent, but it has sensitive dependence on initial conditions.

a) $f : [0,1) \to [0,1)$ is defined by

$$f(x) = \begin{cases} \frac{1}{2}x + \frac{1}{2} & 0 \leq x < 7/1 \\ (2^{n+1} - 4^{n+1} - 2^{-1})(x + 2^{-n} - 2 \cdot 4^{-n-1} - 1) & b_n \\ \frac{1 - 2^{-n-2} - 2 \cdot 4^{-n-3}}{2^{-n-1} - 9 \cdot 4^{-n-2}}(x + 2^{-n} - 2 \cdot 4^{-n-1} - 1) & c_n \leq \end{cases}$$

where $a_n = 1 - 2^{-n} - 4^{-n-1}$, $b_n = 1 - 2^{-n} + 4^{-n-1}$, $c_n = 1 - 2^{-n} + 2 \cdot 4^{-n-1}$ for $n = 1, 2, ...$

b) $g : [0,1] \to [0,1]$ is defined by

$$g(x) = \begin{cases} 3x + \frac{1}{2} & 0 \leq x \leq \frac{1}{15} \\ \frac{6}{127}x + \frac{7}{10} - \frac{2}{635} & \frac{1}{15} < y \leq \frac{1}{2} - \frac{1}{1} \\ 3x + \frac{1}{2} - \frac{5}{2^{n+1}}(2^n - 1) & a_n < x \leq b_n \end{cases}$$

The map $G(x,y) = (f^2(y), g(x))$, is continuous in $[0,1)^2$ since f and g are continuous in $[0,1)$.
Let us consider the trajectory of $(0,0)$, denoted by

$$\{(0,0), (x_1,y_1), (x_2,y_2), \cdots\}$$

with

$$G^{2n}(0,0) = \left(1 - \frac{1}{2^{3n}}, 1 - \frac{1}{2^{3n}}\right), \qquad G^{2n-1}(0,0) = \left(1 - \frac{1}{2^{3n-1}}, 1 - \frac{1}{2^{3n-2}}\right.$$

for $n = 1, 2, ...$
Similarly to the former example, we have

$$DG(0,0) = \begin{pmatrix} 0 & 1/4 \\ 3 & 0 \end{pmatrix} \qquad DG(x_1,y_1) = DG^2(0,0) = \begin{pmatrix} 3/4 & 0 \\ 0 & 3/4 \end{pmatrix}$$

$$DG(x_2,y_2) = DG^3(0,0) = \begin{pmatrix} 0 & 3/4^2 \\ 3^2/4 & 0 \end{pmatrix}$$

and in general we have

$$DG^{2n}(0,0) = \begin{pmatrix} (3/4)^n & 0 \\ 0 & (3/4)^n \end{pmatrix} \qquad DG^{2n-1}(0,0) = \begin{pmatrix} 0 & 3 \cdot (1/4 \\ 3 \cdot (1/4)3^n & 0 \end{pmatrix}$$

Now we compute the eigenvalues of $(DG^n)^t \cdot DG^n$ for $n = 1, 2, \ldots$:

- Eigenvalues of $DG^{2n-1}(0,0)$ are

$$\begin{pmatrix} 0 & \frac{3^n}{4^{n-1}} \\ \frac{3^{n-1}}{4^n} & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{3^{n-1}}{4^n} \\ \frac{3^n}{4^{n-1}} & 0 \end{pmatrix} = \begin{pmatrix} \frac{3^{2n}}{4^{2(n-1)}} & 0 \\ 0 & \frac{3^{2(n-1)}}{4^{2n}} \end{pmatrix}$$

  therefore

$$\frac{1}{2n-1}\left(n\log 3 - (n-1)\log 4\right)$$

- Eigenvalues of $DG^{2n}(0,0)$:
  The related matrix is diagonal and then the we have the double eigenvalue $\left(\frac{3}{4}\right)^n$. Consequently we have

$$\frac{n}{2n}\left(\log 3 - \log 4\right)$$

Therefore

$$\lambda_1(0,0) = \lim_{n\to\infty} \frac{1}{2n}\log(|\mu_1(n,(0,0))|) = \frac{1}{2}\log\left(\frac{3}{4}\right) < 0$$

It is left to prove that the forward trajectory of $(0,0)$ has sensitive dependence on initial conditions, that is, there exists $\epsilon > 0$ such that for every $\delta > 0$ there exists $d((x,y),(0,0)) < \delta$ and $k > 1$ holding $d(G^k(x,y) - G^k(0,0)) > \epsilon$.
Now we compute the distance to the maximum. Taking $\epsilon = 3/8$, then for every $\delta > 0$ there exists (x,y) such that $d((x,y),(0,0)) < \delta$, that is, $x < \delta$, $y < \delta$ and $k \geq 2$ it is hold that $f^k(x) < 1/2$ or $f^k(y) < 1/2$ and by other hand we have $f^k(0) > 7/8$. Therefore

$$d(G^k(0,0) - G^k(x,y)) > \epsilon$$

The previous construction for the two dimensional case on I^2 or $[0,1) \times [0,1) = B$ can be extended to similar constructions on I^n, $\mathbb{B}^n$ or $\mathbb{T}^n$ using general versions of the permutation maps considered in a paper from BL.