Limit cycles and critical periods using higher-order Taylor developments

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Main Reference & Grants

This talk is based on the following joint works

- J. Giné, L. F. d. S. Gouveia, J. Torregrosa. Lower bounds for the local cyclicity for families of centers. J. Differential Equations, 275, 309-331. 2021.
- I. Sánchez-Sánchez, J. Torregrosa. New lower bounds of the number of critical periods in reversible centers. J. Differential Equations, 292, 427-460. 2021.

Grants

PID2019-104658GB-I00
CEX2020-001084-M

2017 SGR 1617

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Quadratic family

Theorem (Bau1953)

The number of limit cycles of small amplitude bifurcating from an equilibrium of monodromic type in the class of polynomial vector fields of degree 2 *is (exactly)* 3.

Theorem (ChiJac1989)

The number of critical periods bifurcating from the origin when perturbing in the class of reversible quadratic systems is (exactly) 2.

Remark. The period function for quadratic potentials is monotonous increasing. The 2 oscillations appear in Loud family.

Local Hilbert Number for cubic and quartic families

Theorem (GinGouTor2021)

The number of limit cycles of small amplitude bifurcating from an equilibrium of monodromic type in the classes of polynomial vector fields of degrees 3 *and* 4 *is (at least)* 12 *and* 21*, respectively.*

Local critical periods for cubics

Theorem (SanTor2021)

Let $a \in \mathbb{R} \setminus \{0\}$. Consider the 1-parameter family of cubic (holomorphic) reversible systems

$$\dot{z} = \mathrm{i}\, z\,(1-z)\,(1-az)\,.$$

The number of critical periods bifurcating from the origin when perturbing in the class of reversible cubic systems is at least 5 for $a \in \{-3/2, -1, -2/3, 1/2, 2\}$ and 4 otherwise.

With the change $z = x + \mathrm{i}\, y$ we can write the above equation as a planar polynomial differential equation:

$$(\dot{x}, \dot{y}) = (-a(3x^2y-y^3)+2(a+1)xy-y,\, a(x^3-3xy^2)-(a+1)x^2+(a+1)y^2+x).$$

Local critical periods for quartics

Theorem (SanTor2021)

Let $a, b \in \mathbb{R}$. Consider the 2-parameter family of quartic (holomorphic) reversible systems

$$\dot{z} = \mathrm{i}\, z\,(1-z)\,(1-az)\,(1-bz)\,.$$

Generically, at least 8 critical periods bifurcate from the origin when perturbing in the class of reversible quartic centers. Moreover, in this perturbation class there exists a point (a, b) such that at least 10 critical periods bifurcate from the origin.

- Implicit Function Theorem
- Extensions of Implicit Function Theorem (Weighted Blow-ups)
- Parallelization and higher order developments
- Working with families instead of fixed vector fields

We perform a change to polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$ on system

$$(\dot{x}, \dot{y}) = (-y + X(x, y), x + Y(x, y)),$$

to obtain

$$(\dot{r}, \dot{\varphi}) = \left(\sum_{k=1}^{\infty} \xi_k(\varphi) r^{k+1}, 1 + \sum_{k=1}^{\infty} \zeta_k(\varphi) r^k, \right)$$

where $\xi_k(\varphi)$ and $\zeta_k(\varphi)$ are homogeneous polynomials in $\sin\varphi$ and $\cos\varphi$ of degree $k+2$. Eliminating the time we have

$$\frac{dr}{d\varphi} = \sum_{k=2}^{\infty} R_k(\varphi) r^k,$$

where $R_k(\varphi)$ are 2π-periodic functions of φ and the series is convergent for all φ and for all sufficiently small r.

The Lyapunov constants

The initial value problem with the initial condition $(r, \varphi) = (\rho, 0)$ has a unique solution

$$r(\varphi) = \rho + \sum_{k=2}^{\infty} u_k(\varphi)\rho^k,$$

which is convergent for all $0 \leq \varphi \leq 2\pi$ and all $\rho < r^*$, for some sufficiently small $r^* > 0$. The coefficients $u_k(\varphi)$ can be determined by simple quadratures.

The **Lyapunov constants** are defined as the coefficients, in ρ, of the return map $\Pi(\rho) = r(2\pi)$.

$$V_k = u_k(2\pi).$$

Which are the properties of the Lyapunov constants?

Limit cycles and centers

The limit cycles are the isolated fixed points of the return map $\Pi(\rho) = r(2\pi)$

or

the isolated zeros of the difference map $\Delta(\rho) = \Pi(\rho) - \rho$.

The origin is a center when $\Pi(\rho) \equiv \rho$ or $\Delta(\rho) \equiv \rho$.

Time function (for centers)

Using the solution $r(\varphi)$ into the second equation of the original equation in polar coordinates, obtaining

$$\dot{\varphi} = \frac{d\varphi}{dt} = 1 + \sum_{k=1}^{\infty} F_k(\varphi)\rho^k.$$

Rewriting this equation as

$$dt = \frac{d\varphi}{1 + \sum_{k=1}^{\infty} F_k(\varphi)\rho^k} = \left(1 + \sum_{k=1}^{\infty} \Psi_k(\varphi)\rho^k\right) d\varphi$$

and integrating, we get

$$t - \varphi = \sum_{k=1}^{\infty} \theta_k(\varphi)\rho^k,$$

where $\theta_k(\varphi) = \int_0^{\varphi} \Psi_k(\psi)\, d\psi$ and all the series converge for $0 \le \varphi \le 2\pi$ and sufficiently small $\rho \ge 0$.

The period constants

The coefficients of the Taylor series, in ρ, of the period function of any closed trajectory define the **period constants**:

$$\mathcal{T}_k = \frac{1}{2\pi}\theta_k(2\pi) = \frac{1}{2\pi}\int_0^{2\pi} \Psi_k(\psi)\, d\psi.$$

Which are the properties of the period constants?

Questions

- How we can use the **constants** for obtaining **oscillations** bifurcating from the origin?
- The number of oscillations depend on the parameters of a **family**?
- How they are **independent**?
- How can we take benefit of perturbing families instead of individual vector fields?
- Implicit Function Theorem and its extensions ..., that is, how can we use the **varieties intersection theory**?

Proving the cubic result (1)

When $a \in \mathbb{R} \setminus \{-1, 0, 1/2, 2\}$, the rank of the linear developments of first four period constants of this system with respect to $(r_{11}, r_{02}, r_{21}, r_{12})$ is 4. After using the Implicit Function Theorem, the period constants take the form

$$T_k = u_k, \text{ for } k = 1, \ldots, 4.$$

Taking $u_1 = u_2 = u_3 = u_4 = 0$ and $r_{03} = u_5$, the fifth and sixth period constants take the form

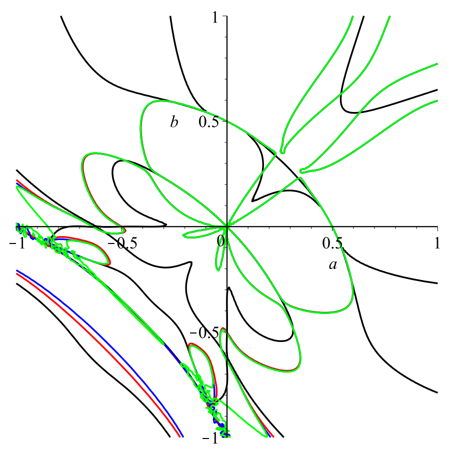
$$\begin{aligned} T_5 &= \frac{5}{24} \frac{P(a)}{3a^2 + 2a + 3} u_5 + u_5^2 \sum_{j=0}^{\infty} f_j(a)\, u_5^j, \\ T_6 &= -\frac{1}{42} \frac{Q(a)}{3a^2 + 2a + 3} u_5 + u_5^2 \sum_{j=0}^{\infty} g_j(a)\, u_5^j, \end{aligned} \tag{1}$$

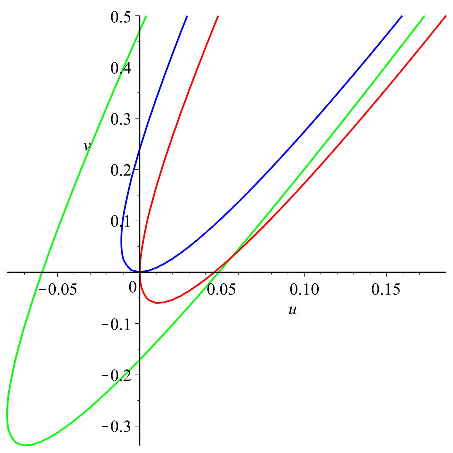
where $P(a) = a^3(a-2)(3a+2)(2a+3)(2a-1)$,
$Q(a) = a^3(a-2)(2a-1)(834a^2 + 1735a + 834)(a+1)^2$, and f_j and g_j are rational functions.

Proving the cubic result (2)

Then, we have 4 critical periods when $P(a) \neq 0$ and 5 when $P(a) = 0$, $P'(a) \neq 0$, and $Q(a) \neq 0$. Then, as $a \neq 0$, the statement follows except for the remaining cases $a \in \{-1, 1/2, 2\}$. These cases need more accurate analysis.

Proving the quartic result





Can we prove that the lines intersect transversally?
(Poincaré-Miranda Theorem + Computer Assisted Proof)