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Formal Integrability and Nilpotent Centers on Center Manifolds in $\mathbb{R}^3$

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Introduction

In this work, we study analytic vector fields X in $\mathbb{R}^3$ with associated differential system in the following form:

$$\begin{aligned}\dot{x} &= y + P(x,y,z),\\ \dot{y} &= \quad Q(x,y,z),\\ \dot{z} &= -\lambda z + R(x,y,z),\end{aligned} \tag{1}$$

where $\lambda \neq 0$ and P, Q, R are analytic functions and $j^1P(0) = j^1Q(0) = j^1R(0) = 0$. For such systems, by the *Center Manifold Theorem*, there exists an invariant bidimensional C^r-manifold tangent to the xy-plane at the origin for every $r \geqslant 1$.

The restriction of this system to a center manifold has a nilpotent singular point at the origin. Henceforth we say that a three-dimensional analytical vector field has a *nilpotent singular point* if its associated differential system can be written as (1).

We study the formal integrability and the center problem for those types of singular points in the monodromic case. Our approach does not require polynomial approximations of the Center Manifold in order to study the center problem. We conclude the work solving the Nilpotent Center Problem for the Generalized Lorenz system.

Monodromy Criterion

Theorem 1 (C^r-Andreev's Theorem) Let X be the vector field associated to the C^r-system, $r \geqslant 3$, given by

$$\begin{aligned}\dot{x} &= y + X_2(x,y),\\ \dot{y} &= Y_2(x,y),\end{aligned} \tag{2}$$

where $X_2, Y_2 \in C^r$, $j^1X_2(0) = j^1Y_2(0) = 0$ and such that the origin is an isolated singular point. Let $y = F(x)$ be the solution of the equation $y + X_2(x,y) = 0$ through $(0,0)$ and consider $f(x) = Y_2(x, F(x))$ and $\Phi(x) = \operatorname{div} X|_{(x,F(x))}$. We can write

$$f(x) = ax^\alpha + O(x^{\alpha+1}),$$

$$\Phi(x) = bx^\beta + O(x^{\beta+1}).$$

for $\alpha < r$. Suppose that $a \neq 0$, then the origin is monodromic if and only if $a < 0, \alpha = 2n-1$ and one of the following conditions holds:

i) $\beta > n-1$ or $j^r\Phi(0) \equiv 0$;

ii) $\beta = n-1$ and $b^2 + 4an < 0$;

The positive integer n in the statements of Theorem 1 plays an important role in the study of nilpotent monodromic singular points. So we define the *Andreev number* of a nilpotent singular point by the number n in function $f(x) = ax^{2n-1} + O(x^{2n})$. The Andreev number is invariant by analytical and formal orbital equivalence, i.e. via analytical and formal diffeomorphisms and time rescalings [1].

Consider the following representation of analytic system (1):

$$\begin{aligned}\dot{x} &= \quad y + \sum_{j+k+l\geqslant 2} a_{jkl}x^jy^kz^l,\\ \dot{y} &= \qquad \sum_{j+k+l\geqslant 2} b_{jkl}x^jy^kz^l,\\ \dot{z} &= -\lambda z + \sum_{j+k+l\geqslant 2} c_{jkl}x^jy^kz^l.\end{aligned} \tag{3}$$

Proposition 1 The origin is a nilpotent monodromic singular point with Andreev number 2 on a center manifold of system (3) if and only if $b_{200} = 0$ and

$$\frac{b_{101}c_{200}}{\lambda} < -\frac{(2a_{200} - b_{110})^2}{8} - b_{300}. \tag{4}$$

Moreover, if $2a_{200} + b_{110} \neq 0$, the restricted system satisfies the monodromy condition $\beta = n-1$ in Theorem 1.

Proposition 2 For C^r planar systems having an isolated monodromic nilpotent singular point at the origin, with Andreev number n such that $2n - 1 < r$, the monodromy conditions (i) and (ii) in Theorem 1 are invariant by local diffeomorphisms.

Belitskii Formal Normal Form

We first look at the Normal Form theory to search the most practical normal forms for system (1). Using Theorem 5.1 in [2] we obtain the following result:

Theorem 2 (Nilpotent Normal Form in $\mathbb{R}^3$) For system (1) having a nilpotent singular point at the origin, there exist a formal change of variables that transforms it into the formal normal form

$$\begin{aligned}\dot{x} &= y + xP_1(x),\\ \dot{y} &= Q_2(x) + yP_1(x),\\ \dot{z} &= -\lambda z + zR_1(x).\end{aligned} \tag{5}$$

for which $P_1(0) = j^1Q_2(0) = R_1(0) = 0$.

We were not able to prove that the normal form (5) is analytic, i.e. that the series P_1, Q_2 and R_1 are convergent. That does not mean that this normal form is not useful. We remark that the above normal form has $z = 0$ as an invariant surface which is a center manifold and the first two components are decoupled from the third.

The Belitskii normal form for planar systems having a nilpotent singular point is

$$\begin{aligned}\dot{x} &= y + xP_1(x),\\ \dot{y} &= Q_2(x) + yP_1(x),\end{aligned} \tag{6}$$

where $j^1Q_2(0) = P_1(0) = 0$, which is very similar to its three-dimensional counterpart.

Formal Integrability

Theorem 3 Consider a vector field X associated to system (1) having a nilpotent singular point. Then there exists a formal series $H(x,y,z) = y^2 + \sum_{n\geqslant 3} H_n(x,y,z)$ such that $XH = \sum_{n\geqslant 4} \omega_n x^n$.

Theorem 4 Let X be the vector field associated to system (1) having a nilpotent singular point and H be a formal series as in Theorem 3. If there exists $n \in \mathbb{N}$ such that $j^{2n}XH(0) = \omega_{2n}x^{2n}$ with $\omega_{2n} \neq 0$, then the origin cannot be a center on the center manifold.

Lemma 1 If H is a formal first integral for the normal form (5), then $\dfrac{\partial H}{\partial z} \equiv 0$, that is $H = H(x,y)$.

The previous lemma lets us conclude that formal integrability of the normal form (5) is essentially formal integrability of the planar normal form (6). Hence, we proceed to study the formal integrability of the formal system (6).

One of the consequences of Lemma 1 is that if system (1) is formally integrable, then it has a formal first integral H such that $j^2H(0) = y^2$. In fact, since (1) is transformed into (5) via near-identity changes of variables and the formal integrability of system (1) is equivalent to the formal integrability of system (5), it is enough to verify this statement for system (6). But this is a known result [3]. Therefore, the quantities ω_n in Theorem 3 present obstructions for the system (1) to be analytically or formally integrable. For integrable systems, the monodromic singular point must be a center, since in this case the restricted system is also integrable. However not all nilpotent centers are formally integrable. For instance, consider the following system:

$$\begin{aligned}\dot{x} &= y + x^2,\\ \dot{y} &= -x^3,\\ \dot{z} &= -\lambda z.\end{aligned} \tag{7}$$

which has $z = 0$ as a center manifold and its restriction is a time-reversible system which implies that the origin is a center on the center manifold. If we try to construct a formal first integral $H(x,y,z)$ for system we obtain $\omega_5 = 2$. Note that the obstruction is a coefficient of odd power of x in XH. If it was an even power, we would not have a nilpotent center.

Lemma 2 Consider system (6) having a monodromic singular point satisfying monodromy condition $\beta = n-1$ in Theorem 1. Then it is not formally integrable.

Theorem 5 Consider system (1) having a monodromic singular point such that its restriction to a center manifold satisfies monodromy condition $\beta = n-1$ in Theorem 1. Then it cannot admit formal first integral.

Theorem 6 Suppose that the origin of system (1) is monodromic with odd Andreev number n and satisfies the monodromy condition $\beta = n-1$. Then the origin cannot be a center on a center manifold.

Theorem 7 Consider system (5) having a monodromic singular point. If it admits formal first integral H, then either $P_1(x) \equiv 0$ or $m = 2sn - 1$ for some $s \in \mathbb{N}$.

Generalized Lorenz system

The Generalized Lorenz system is one of the most studied three-dimensional systems in the literature since its dynamics are very rich. Among the particular cases of the Generalized Lorenz systems are the Lü and Chen systems. Its expression is given by

$$\begin{aligned}\dot{x} &= a(y-x),\\ \dot{y} &= bx + cy - xz,\\ \dot{z} &= dz + xy.\end{aligned} \tag{8}$$

We consider $ad \neq 0$ for the singular points of system (8) to be isolated. For the origin to be a nilpotent singular point, we must have $b + c = 0$ and $c = a$. By means of the coordinate change $\bar{x} = y$, $\bar{y} = a(y-x)$, $\bar{z} = z$, dropping the bars, system (8) becomes

$$\begin{aligned}\dot{x} &= y - xz + \tfrac{1}{a}yz,\\ \dot{y} &= -axz + yz,\\ \dot{z} &= dz + x^2 - \tfrac{1}{a}xy.\end{aligned} \tag{9}$$

Theorem 4 is powerful enough to solve the Nilpotent Center Problem for the above system. We compute the quantities ω_n (Theorem 3) for system (9) and obtain the first non-zero one:

$$\omega_6 = -\frac{2\,a\,(2\,a + d)}{3d^3}.$$

Thus, by Theorem 4, the origin can only be a nilpotent center on a center manifold if $d = -2a$. Under this condition, by Proposition 1, the origin is monodromic with Andreev number 2. Moreover, it satisfies monodromy condition $\beta > n-1$ from Theorem 1.

For $d = -2a$, the function $V(x,y,z) = x^2 - \frac{2xy}{a} + \frac{y^2}{a^2} - 2az$ defines an invariant surface $V \equiv 0$ for (9) which is tangent to the xy-plane, thus, it is a center manifold for system (9). Moreover, the restriction of the system to this center manifold is a Hamiltonian system. Thus, the origin is a nilpotent center on a center manifold. We conclude:

Theorem 8 The origin of the Generalized Lorenz system (8) is a nilpotent center on a center manifold if and only if $b = -a$, $c = a$, $d = -2a$.

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