Ulam-Hyers stability and exponential dichotomy

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Content

1 Introduction

2 Finite dimensional linear systems

3 Linear equations

4 Nonlinear equations

1 Introduction

2 Finite dimensional linear systems

3 Linear equations

4 Nonlinear equations

Ulam 1940 and Hyers 1941

Ulam-Hyers stability of a given equation is its property of having a solution sufficiently near each approximate solution.

Stan Ulam proposed in 1940 in a talk at the University of Wisconsin to study this type of stability for the linear functional equation

$$f(x+y) = f(x) + f(y),$$

where the unknown f is a map between Banach spaces. A positive answer was given by D. H. Hyers one year later in

D.H. Hyers, On the stability of the linear functional equation, Proc. Nat. Acad. Sci. USA 27 (1941), 222-224.

Definitions for linear ODEs

Let $A \in C(\mathbb{R}, \mathcal{L}(\mathbb{C}^n))$. We say that

$$x' = A(t)x$$

is *Ulam-Hyers stable* when there exists a constant $m > 0$ such that, for any $\varepsilon > 0$ and any $\varphi \in C^1(\mathbb{R}, \mathbb{C}^n)$ with

$$|\varphi'(t) - A(t)\varphi(t)| \leq \varepsilon, \quad t \in \mathbb{R},$$

there exists $\psi \in C^1(\mathbb{R}, \mathbb{C}^n)$ a solution of $x' = A(t)x$, such that $(\varphi - \psi) \in C_b(\mathbb{R}, \mathbb{C}^n)$ and

$$|\varphi - \psi|_\infty \leq m\varepsilon.$$

We say that the equation $x' = A(t)x$ is *Ulam-Hyers stable with uniqueness* when, for a given φ as above, there exists a unique ψ.

Definitions for linear ODEs

Let $a_1, \ldots, a_n \in C(\mathbb{R}, \mathbb{C})$. We say that

$$x^{(n)} + a_1(t)x^{(n-1)} + \ldots + a_n(t)x = 0$$

is *Ulam-Hyers stable* on the time interval $\mathbb{R}$ if there exists $m > 0$ such that, for any $\varepsilon > 0$, and any $\varphi \in C^n(\mathbb{R}, \mathbb{C})$ for which

$$|\varphi^{(n)}(t) + a_1(t)\varphi^{(n-1)}(t) + \ldots + a_n(t)\varphi(t)| \leq \varepsilon, \quad t \in \mathbb{R},$$

there exists $\psi \in C^n(\mathbb{R}, \mathbb{C})$ solution of the equation such that $(\varphi - \psi) \in C_b(\mathbb{R}, \mathbb{C}^n)$ and

$$|\varphi - \psi|_\infty \leq m\varepsilon.$$

We say that the equation $x' = A(t)x$ is *Ulam-Hyers stable with uniqueness* when, for a given φ as above, there exists a unique ψ.

Let $\lambda \in \mathbb{R}$ and $x' = \lambda x$.

Claudi Alsina, Roman Ger, On some inequalities and stability results related to the exponential function, Journal of Inequalities and Applications, 2 (1998), 373-380.

Theorem

If $\lambda = 0$ then $x' = 0$ is not UH-stable.
If $\lambda \neq 0$ then $x' = \lambda x$ is UH-stable with uniqueness and the best constant is $m = \frac{1}{|\lambda|}$.

$\lambda = 0$: $\varphi(t) = \varepsilon t + c_1$, $\psi(t) = c_2$ implies
$\varphi(t) - \psi(t) = \varepsilon t + c_1 - c_2$.

$x' = \lambda x$ with $\lambda \neq 0$

For any $g \in C_b(\mathbb{R})$ there exists a unique $u \in C_b(\mathbb{R})$ solution of

$$u' - \lambda u = g(t).$$

Moreover, this function is

$$u(t) = e^{\lambda t} \int_{\operatorname{sgn}(\lambda)\infty}^{t} g(s) e^{-\lambda s} ds$$

and

$$|u|_\infty \leq \frac{1}{|\lambda|} |g|_\infty.$$

Let φ be an ε-solution, let $g(t) = \varphi'(t) - \lambda \varphi(t)$. Then $|g|_\infty \leq \varepsilon$. Take $\psi = u - \varphi$.

1 Introduction

2 Finite dimensional linear systems

3 Linear equations

4 Nonlinear equations

Property (M)

We say that the equation $x' = A(t)x$ has property (M) when $x' = A(t)x + g(t)$ has a bounded solution for any $g \in C_b(\mathbb{R}, \mathbb{C}^n)$.

We say that the equation $x' = A(t)x$ has property (M) with uniqueness when the bounded solution is unique.

Theorem

$x' = A(t)x$ is Ulam-Hyers stable (with uniqueness) if and only if $x' = A(t)x$ has property (M) (with uniqueness).

Proof

First assume that $x' = A(t)x$ is Ulam-Hyers stable.

Let g be a bounded function and φ be a solution of $x' = A(t)x + g$.

Then φ is an $|g|_\infty$-solution of $x' = A(t)x$.

We have that $(\varphi - \psi)$ is a bounded solution of $x' = A(t)x + g$.

Proof

Now assume that $x' = A(t)x$ has property (M).

Denote with R_1 the set of initial values $\eta \in \mathbb{C}^n$, for which the solution of the IVP $x' = A(t)x$, $x(0) = \eta$ is bounded.

Additionally let R_2 be the supplementary space of R_1, i.e., $\mathbb{C}^n = R_1 \oplus R_2$.

A theorem in [W.A. Coppel, Stability and asymptotic behavior of differential equations, Heath, 1965] states that there exists $m > 0$, and for each bounded g there exists a unique bounded solution u of $X' = A(t)X + g$ with $u(0) \in R_2$. Moreover, $|u|_\infty \leq m|g|_\infty$. From here the UH stability of $X' = A(t)X$ is obtained.

A practical result

Lemma

Let the matrix $Q(t)$ be such that $Q(t)$ is invertible for any $t \in \mathbb{R}$ and both Q and Q^{-1} are bounded on $\mathbb{R}$. Consider $y' = B(t)y$ obtained from $x' = A(t)x$ by the change of variable $y = Q(t)x$. Then we have that $x' = A(t)x$ is UH stable (with uniqueness) if and only if $y' = B(t)y$ is UH stable (with uniqueness).

Remark. If $Q(t)$ is periodic, then both Q and Q^{-1} are bounded on $\mathbb{R}$.

Linear systems with constant coefficients

Theorem

If $Re(\lambda) \neq 0$ for each $\lambda \in \sigma(A)$ then $x' = Ax$ has property (M) with uniqueness.
Otherwise, $x' = Ax$ does not have property (M).

Proof. It is possible to consider only the case when A is a real Jordan block. Let λ be its eigenvalue.
First assume that $Re(\lambda) < 0$. Then there exist $M, \omega > 0$ such that

$$||e^{tA}|| \leq Me^{-\omega t}, \quad t \in [0, \infty).$$

Proof

For each g bounded the unique bounded solution of $X' = Ax + g$ is

$$u(t) = \int_{-\infty}^{t} e^{(t-s)A} g(s) ds.$$

Moreover, we have

$$|u|_\infty \leq \frac{M}{\omega} |g|_\infty.$$

Proof

Assume that $Re(\lambda) = 0$.

When $\lambda = 0$, due to the fact that A is a Jordan block, one of the equations of the system $X' = AX$ has the form $x' = 0$. We choose $g \in C_b(\mathbb{R}, \mathbb{R}^n)$ such that the corresponding equation of the system $u' = Au + g(t)$ is $x' = 1$. Then any solution of $u' = Au + g(t)$ must be unbounded.

When $\lambda = \pm i\beta$, two equations of the system $X' = AX$ must be of the form $x' = -\beta y$, $y' = \beta x$. We have that any solution of the system $x' = -\beta y$, $y' = \beta x + \cos(\beta t)$ is unbounded.

Linear systems with periodic coefficients

Theorem

If each of the characteristic multipliers of the periodic system $x' = A(t)x$ is not on the unit circle then $x' = A(t)x$ has property (M) with uniqueness.
Otherwise $x' = A(t)x$ does not have property (M).

Proof. By Floquet theory, there exist $Q(t)$ a periodic matrix and R a constant matrix such that the linear change of variables $x = Q(t)y$ gives $y' = Ry$.

Moreover, the condition on the characteristic multipliers of $x' = A(t)x$ is equivalent to $Re(\lambda) \neq 0$ for each $\lambda \in \sigma(R)$.

Uniform exponential dichotomy

Let $E(t)$ be the principal matrix solution of $x' = A(t)x$ and

$$U(\theta, \tau) = E(\theta)E(\tau)^{-1}.$$

We say that $x' = A(t)x$ admits a uniform exponential dichotomy on $\mathbb{R}$ if there exists a family of linear bounded projectors $P(t)$ for $t \in \mathbb{R}$ with $P(\theta)U(\theta, \tau) = U(\theta, \tau)P(t)$ for $\theta \geq \tau$, and there exists $M, \omega > 0$ such that, for each $\theta \geq \tau$,

$$||U(\theta, \tau)P(\tau)|| \leq Me^{-\omega(\theta-\tau)} \quad \text{and} \quad ||U(\theta, \tau)^{-1}(I - P(\theta))|| \leq Me^{-\omega(\theta-\tau)}.$$

I. Assume that $Re(\lambda) \neq 0$ for each $\lambda \in \sigma(A)$. Then $x' = Ax$ admits a uniform exponential dichotomy.

II. Assume that each of the characteristic multipliers of the periodic system $x' = A(t)x$ is not on the unit circle. Then $x' = A(t)x$ admits a uniform exponential dichotomy.

Uniform exponential dichotomy and property (M)

Theorem (Coppel)

Assume that the system $x' = A(t)x$ has bounded coefficients. We have that $x' = A(t)x$ admits a uniform exponential dichotomy if and only if it has property (M) with uniqueness.

Corollary

Assume that the system $x' = A(t)x$ has bounded coefficients. We have that $x' = A(t)x$ is Ulam-Hyers stable with uniqueness if and only if it admits a uniform exponential dichotomy.

Nonuniqueness

Theorem

Assume that $X' = A(t)X$ has property (M) and that it has nonnull bounded solutions. Then $\lim_{t\to\pm\infty} t^k v(t) = 0$ for any $k \in \mathbb{N}$ and any bounded solution v of $X' = A(t)X$.

Proof. There exists a sequence of bounded functions $(u_k)_{k\geq 0}$ defined by

$$u_0 = v, \quad u'_{k+1} = A(t)u_{k+1} + u_k \quad k \geq 0.$$

Now define a sequence of functions $(v_k)_{k\geq 0}$ explicitly by $v_0 = v$ and

$$v_k = u_k + \Sigma_{j=1}^{k-1}(-1)^j \frac{t^j}{j!} u_{k-j} + (-1)^k \frac{t^k}{k!} v.$$

Proof

By direct computations we get that, for each $k \geq 0$, v_k is a solution of $X' = A(t)X$.

Thus the $n+1$ functions v, v_1, ... , v_{n-1}, v_{n+j} are linearly dependent for each $j \geq 0$. Then there are constants c_0, c_1, ... , c_{n-1}, c_n such that

$$c_0 v + c_1 v_1 + \ldots + c_{n-1} v_{n-1} + c_n v_{n+j} = 0,$$

where

$$v_k = u_k + \Sigma_{j=1}^{k-1} (-1)^j \frac{t^j}{j!} u_{k-j} + (-1)^k \frac{t^k}{k!} v.$$

1 Introduction

2 Finite dimensional linear systems

3 Linear equations

4 Nonlinear equations

Linear equations with constant coefficients

T. Miura, S. Miyajima, S.E. Takahasi, Hyers-Ulam stability of linear differential operator with constant coefficients, Math. Nachr. 258 (2003) 90-96.

Theorem

The linear equation with constant coefficients of arbitrary order is Ulam-Hyers stable on $\mathbb{R}$ if and only if its characteristic polynomial has only roots with $\Re(\lambda) \neq 0$.

$x^{(n)} + a_1(t)x^{(n-1)} + \cdots + a_n(t)x = 0$ and its equivalent system $X' = A(t)X$ with $X = (x, x', ..., x^{(n-1)})$.

Bing Xu, Janusz Brzdek, and Weinian Zhang, Fixed point results and the Hyers-Ulam stability of linear equations of higher orders, Pacific Journal of Mathematics, 273(2):483498, 2015.

Linear equations with periodic coefficients

$$Lx = x^{(n)} + a_1(t)x^{(n-1)} + \cdots + a_n(t)x.$$

Theorem

The linear equation with periodic coefficients $Lx = 0$ is Ulam-Hyers stable if and only if its equivalent system $X' = A(t)X$ with $X = (x, x', ..., x^{(n-1)})$ is Ulam-Hyers stable.

Proof.
I. If the corresponding system $X' = A(t)X$ is Ulam-Hyers stable then the equation is Ulam-Hyers stable.
II. Now consider that the equation is Ulam-Hyers stable. Then for any bounded g there exists a bounded solution of $Lx = g$.

Proof

Assume, by contradiction that the system is not Ulam-Hyers stable. Our aim is to construct a g such that any solution of $Lx = g$ is unbounded.

We have that the linear system has a characteristic multiplier on the unit circle. Then $\lambda_1 = i\beta_1$ is a characteristic exponent, whose characteristic multiplier has modulus one. There exists a periodic function $f(t)$ such that

$$q(t) = f(t)e^{i\beta_1}$$

satisfies $Lq = 0$. Of course, q is bounded.

Since the mappings $q, tq, \ldots, t^{n+1}q$ are linearly independent, there is a smallest $\mu \geq 1$ such that $L(t^{\mu}q) \neq 0$, hence by definition

$$L(q) = L(tq) = \ldots = L(t^{\mu-1}q) = 0.$$

Proof

Now let us define the differential operators

$$\mathcal{L}_0 = L, \quad \mathcal{L}_i(x) = \sum_{k=i}^{n} (n-k+1)\cdots(n-k+i) a_{k-i} x^{(n-k)}, \quad i = \overline{1, n}.$$

Define

$$g = \mathcal{L}_\mu(q).$$

Looking at the expression of $\mathcal{L}_\mu$ we can see that $\mathcal{L}_\mu(q)$ is bounded, since all derivatives of q are bounded, and so are the coefficients of $\mathcal{L}_\mu$. Therefore $g \in C_b(\mathbb{R}, \mathbb{C})$.

We showed that

$$g = \mathcal{L}_\mu(q) = L(t^\mu q).$$

Hence any solution of $L(x) = g$ is of the form $t^\mu q x_0 + \tilde{x}(t)$, where $\tilde{x}$ is a solution of $L(x) = 0$.

Then we showed that $t^\mu q x_0 + \tilde{x}$ is unbounded.

1 Introduction

2 Finite dimensional linear systems

3 Linear equations

4 Nonlinear equations

Perturbations with small Lipschitz constant

Theorem

Let $x' = A(t)x$ be uniformly exponentially dichotomic and $f(t,x)$ be globally Lipschitz with respect to x, with a sufficiently small Lipschitz constant. Then $x' = A(t)x + f(t,x)$ is Ulam-Hyers stable with uniqueness.

Proof. Use the Banach contractions fixed point theorem.

A local result

Johanna D. Garcia-Saldana, Armengol Gasull, A theoretical basis for the Harmonic Balance Method, J. Differential Equations, 254 (2013), 67-80.

Theorem

Let $\varphi(t)$ be an approximate T-periodic solution of the T-periodic equation $x' = f(t, x)$. Assume that it is "non-critical". Then there exists a T-periodic solution of $x' = f(t, x)$, unique in a given region, and sufficiently close to φ.

A.B., Ulam-Hyers stability and exponentially stable evolution equations in Banach spaces, Carpathian Journal of Mathematics, 37 (2021).

A.B. and György Tőtős, Characterization of Ulam-Hyers stability of linear differential equations with periodic coefficients, submitted.

A.B., Ulam-Hyers stability and exponentially dichotomic evolution equations in Banach spaces, EJQTDE, accepted.

Thank you for your attention.