Perturbation Theory to any order and Hilbert's 16th problem on period annuli

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Introduction

We are interested in quadratic perturbations of the following special reversible Lotka-Volterra quadratic system

$$X_0 : \begin{cases} \dot{x} = -y - x^2 + y^2, \\ \dot{y} = x - 2xy \end{cases} \qquad (1)$$

The double Lotka-Volterra has a double center

This equivalently written in coordinates $z = x + iy$, $\bar{z} = x - iy$ as

$$\frac{dz}{dt} = iz - z^2. \tag{2}$$

This implies that the vector field X_0 has a center at $z = 0$ and $z = i$, that is to say at the origin $(x, y) = (0, 0)$ and at $(x, y) = (0, 1)$, see fig.1. The period of the orbits is

$$T = \int dt = \oint \frac{dz}{iz - z^2} = 2\pi$$

hence the two centers are isochronous (the orbits have a constant period).

Main result

We are interested in the limit cycles which an arbitrary quadratic deformation of (1) can have. The limit cycles on a finite distance from the origin form two nests, containing either the focus close to $(0,0)$, or the focus close to $(0,1)$. We denote their number by i and j.
The main result of the paper is easy to formulate: *the possible distributions* (i,j) *of limit cycles are those, for which* $i+j\leq 2$.
Although the above result is simple, it hides several difficulties, which were not resolved until recently. It illustrates some recent developments of the bifurcation theory of planar vector fields of infinite co-dimension.

First integral

The system (1) has, as suggested by (2), a first integral

$$H = \frac{x^2+y^2}{2y-1} = \frac{x^2+(y-1)^2}{2y-1} + 1 \tag{3}$$

It induces a polynomial foliation on $\mathbb{R}^2$ (or $\mathbb{C}^2$) defined by

$$(1-2y)^2 dH = 0. \tag{4}$$

Perturbation setting

An arbitrary quadratic perturbation of (4) or (1), can be written in one of the following alternative forms

$$\frac{1}{2}(1-2y)^2 dH + \sum_{0\le i,j\le 2} (a_{ij}x^i y^j dy + b_{ij}x^i y^j dx) = 0 \tag{5}$$

or

$$X_{a,b} : \begin{cases} \dot{x} = -y - x^2 + y^2 + \sum_{0\le i,j\le 2} a_{ij}x^i y^j, \\ \dot{y} = \quad x - 2xy - \sum_{0\le i,j\le 2} b_{ij}x^i y^j \end{cases} \tag{6}$$

Perturbation setting in complex variables

In coordinates $z = x + iy, \bar{z} = x - iy$, and up to an affine transformation of $\mathbb{R}^2$ and a scaling of time, each of the above systems can be written in the following normal form

$$\dot{z} = (\lambda_1 + i)z + Az^2 + Bz\bar{z} + C\bar{z}^2, B, C \in \mathbb{C}, \lambda_1 \in \mathbb{R} \tag{7}$$

where

$$A = -1,\ B = \lambda_2 + i\lambda_3, C = \lambda_4 + i\lambda_5,\ \lambda_i \in \mathbb{R}. \tag{8}$$

and $\lambda_1, \lambda_2, \ldots, \lambda_5$ are small real constants.

Complex perturbation setting

We obtain finally the vector field

$$X_\lambda : \begin{cases} \dot{x} = -y - x^2 + y^2 + \lambda_1 x + \lambda_2(x^2 + y^2) + \lambda_4(x^2 - y^2) + 2\lambda_5 xy, \\ \dot{y} = x - 2xy + \lambda_1 y + \lambda_3(x^2 + y^2) + \lambda_5(x^2 - y^2) - 2\lambda_4 xy. \end{cases} \tag{9}$$

to be studied in this talk.

Change of parameters

Thus, to obtain from (6), the normal form (9), we have to substitute

$$\begin{aligned} a_{10} &= \lambda_1, & b_{01} &= -\lambda_1 \\ a_{20} &= \lambda_2 + \lambda_4, & b_{20} &= -\lambda_3 - \lambda_5 \\ a_{02} &= \lambda_2 - \lambda_4, & b_{02} &= -\lambda_3 + \lambda_5 \\ a_{11} &= 2\lambda_5, & b_{11} &= 2\lambda_4 \end{aligned}$$

and $a_{00} = b_{00} = a_{01} = b_{10} = 0$.

Final form

The foliation underlying the vector field X_λ takes the form

$$\omega = \frac{1}{(1-2y)^2}\sum_{0\leq i,j\leq 2}(a_{ij}x^i y^j dy + b_{ij}x^i y^j dx) = \sum_{i=1}^{5}\lambda_i\omega_i$$
$$\omega_1 = \frac{xdy-ydx}{(2y-1)^2}, \omega_2 = \frac{(x^2+y^2)dy}{(2y-1)^2}, \omega_3 = -\frac{x^2+y^2}{(2y-1)^2}dx$$
$$\omega_4 = \frac{(x^2-y^2)dy+2xydx}{(2y-1)^2}, \omega_5 = \frac{2xydy-(x^2-y^2)dx}{(2y-1)^2}.$$

Poincaré first return map and the Bautin ideals

Let $\mathcal{P}(h)$ be the Poincaré first return map associated to one of the foci of (6), which are close to $(0,0)$ and $(0,1)$ for a_{ij}, b_{ij} sufficiently small. Here h is as usual the restriction of the first integral H of the non-perturbed system on a transversal open segment through the focus. It is easily seen that $\mathcal{P}(h)$ is analytic both in h and the parameters a_{ij}, b_{ij}, provided that the deformation is small and h is close to the critical value of H. Expanding the displacement map

$$\mathcal{P}(h) - h$$

in a power series in h

$$\mathcal{P}(h) - h = \sum_{k=0}^{\infty} p_k(\lambda) h^k$$

we consider the ideal $\mathcal{B} =< p_k(\lambda) >\subset \mathbb{R}\{x, y\}$ generated by the coefficients of h^k. The fundamental fact about this Noetherian ideal is, that it is *polynomially generated*.
The main advantage of the form (7) is that its Bautin ideal is known and relatively simple, which is not the case of (6). For this reason the forms (7) and (9) will be used from now on.

Pair of Bautin ideals

We denote $\mathcal{B}_1, \mathcal{B}_2$ the local Bautin ideals associated to $(0, 0)$ and $(0, 1)$, localised at $\lambda = 0$. Our first result is the explicit form of the generators of $\mathcal{B}_1, \mathcal{B}_2$ in the parameter space $\mathbb{R}\{\lambda_1, \dots, \lambda_6\}$. It follows from this result, that the irreducible algebraic set of quadratic systems of Lotka-Volterra type $\mathcal{L}(1, 1, 1)$, has a self-intersection at the "point" X_0. The two local branches of $\mathcal{L}(1, 1, 1)$ near the point X_0 are interchanged by the involution on the parameter space, induced by the affine involution $z \mapsto i - z$.

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The pair of Bautin ideals

Theorem

The vector field X_λ has, for small parameters λ_i, two foci close to $(0,0)$ and $(0,1)$. The respective Bautin ideals $\mathcal{B}_1, \mathcal{B}_2$ are given by

$$\mathcal{B}_1 = < \lambda_1, \lambda_3, \lambda_2\lambda_5 > \tag{10}$$

$$\mathcal{B}_2 = < \lambda_1 + \lambda_3 + \lambda_1\lambda_2, \lambda_5, \lambda_3\lambda_4 > . \tag{11}$$

The zero locus of $\mathcal{B}_1$ has two irreducible components corresponding to systems or reversible or Lotka-Volterra type :

$$\lambda_1 = \lambda_3 = \lambda_5 = 0 \quad \text{(reversible component)} \tag{12}$$

$$\lambda_1 = \lambda_2 = \lambda_3 = 0 \quad \text{(Lotka-Volterra component)} \tag{13}$$

with a similar structure of the isomorphic zero locus of $\mathcal{B}_2$

$$\lambda_1 = \lambda_3 = \lambda_5 = 0 \quad \text{(reversible component)} \tag{14}$$

$$\lambda_1 + \lambda_3 + \lambda_1\lambda_2 = \lambda_4 = \lambda_5 = 0 \quad \text{(Lotka-Volterra component)} \tag{15}$$

First part of the proof of the theorem

The proof goes back to Dulac (1908). Indeed, following Zoladek and Iliev we deduce that $\mathcal{B}_1$ is generated by λ_1 and the focal values v_3, v_5, v_7 where

$$v_3 = 2\pi \mathrm{Im} AB \tag{16}$$

$$v_5 = \frac{2}{3}\mathrm{Im}[(2A+\bar{B})(A-2\bar{B})\bar{B}C] \tag{17}$$

$$v_7 = \frac{5}{4}(|B|^2-|C|^2)\mathrm{Im}[(2A+\bar{B})\bar{B}^2 C]. \tag{18}$$

According to (8) $A=-1$ and $v_3 = -2\pi\lambda_3$. Assuming that $\lambda_3 = 0$ we have

$$v_5 = \frac{2}{3}\mathrm{Im}[(-2+\lambda_2)(-1-2\lambda_2)\lambda_2(\lambda_4+i\lambda_5)] \tag{19}$$

$$= \frac{2}{3}(-2+\lambda_2)(-1-2\lambda_2)\lambda_2\lambda_5 \tag{20}$$

and therefore locally $< v_3, v_5 > = < \lambda_3, \lambda_2\lambda_5 >$ and v_7 is generated by v_3, v_5, which proves (10).

Second part of the proof

The proof of (11) will be done in two steps.

1. First, we prove that the center set of the second center (the variety of the ideal $\mathcal{B}_2$) is defined by (14), (15). For this, we show that when the vector field satisfies (14), (15), then it has a first integral analytic near $(0, 1)$ and hence has a center.
2. Second, we show that the ideal $\mathcal{B}_2$ is radical. For this we use the information obtained from the computation of the first and the second Melnikov functions, and a version of Nakayama lemma, as suggested by Briskin-Roytfarf-Yomdin.

The zero set of the ideal $\mathcal{B}_2$

In what follows we assume that $\lambda_5 = \lambda_4 = 0$. The equation (7) takes the form

$$\dot{z} = (\lambda_1 + i)z - z^2 + Bz\bar{z} = z(\lambda_1 + i + B\bar{z}) \quad (21)$$

where $B = \lambda_2 + i\lambda_3 \in \mathbb{C}$ and $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$. The above equation can be eventually integrated as follows. Consider the underlying foliation defined by

$$(1 - i\lambda_1 + iz - iB\bar{z})zd\bar{z} + (1 + i\lambda_1 - i\bar{z} + i\bar{B}z)\bar{z}dz = 0. \quad (22)$$

It is integrable if and only if it has at least three invariant lines intersecting at singular points of the foliation.

The zero set is a center

Indeed, the following two lines are obviously invariant

$$z = x + iy = 0, \bar{z} = x - iy = 0$$

and let the third one be

$$\alpha z + \bar{\alpha}\bar{z} + 1 = 0, \alpha \in \mathbb{C}.$$

This implies on its turn an ansatz for the first integral as follows :

$$H = z^{1+i\lambda_1}\bar{z}^{1-i\lambda_1}(\alpha z + \bar{\alpha}\bar{z} + 1)^{\beta}, \beta \in \mathbb{R}.$$

The polynomial foliation $d \log H = 0$ is

$$(1 + i\lambda_1)\frac{dz}{z} + (1 - i\lambda_1)\frac{d\bar{z}}{\bar{z}} + \beta\frac{d(\alpha z + \bar{\alpha}\bar{z} + 1)}{\alpha z + \bar{\alpha}\bar{z} + 1} = 0$$

or equivalently

$$\begin{aligned} (\alpha z + \bar{\alpha}\bar{z} + 1)[(1 + i\lambda_1)\bar{z}dz + (1 - i\lambda_1)zd\bar{z}] + \beta z\bar{z}d(\alpha z + \bar{\alpha}\bar{z} + 1) = 0 \\ (1 + i\lambda_1)\bar{z}dz + (1 - i\lambda_1)zd\bar{z} \\ +(1 + i\lambda_1)(\alpha z + \bar{\alpha}\bar{z})\bar{z}dz + (1 - i\lambda_1)(\alpha z + \bar{\alpha}\bar{z})zd\bar{z} + \beta z\bar{z}d(\alpha z + \bar{\alpha}\bar{z}) = 0 \end{aligned}$$

and finally

$$\begin{aligned} (1 - i\lambda_1)zd\bar{z} + [(1 - i\lambda_1)\alpha z^2 + (\bar{\alpha}(1 - i\lambda_1) + \beta\bar{\alpha})z\bar{z}]d\bar{z} \\ +(1 + i\lambda_1)\bar{z}dz + [(1 + i\lambda_1)\alpha\bar{z}^2 + (\alpha(1 + i\lambda_1) + \beta\alpha)\bar{z}z]dz = 0. \end{aligned}$$

existence of an integral

Comparing this to (22) we impose

$$(1 - i\lambda_1)\alpha = i$$
$$\bar{\alpha}(1 - i\lambda_1 + \beta) = -iB$$

where $B = \lambda_2 + i\lambda_3$. Therefore

$$1 - i\lambda_1 + \beta = (\lambda_2 + i\lambda_3)(1 + i\lambda_1)$$

and finally

$$1 + \beta = \lambda_2 - \lambda_1\lambda_3$$
$$-\lambda_1 = \lambda_1\lambda_2 + \lambda_3.$$

The conclusion is that if

$$\lambda_1 + \lambda_3 + \lambda_1\lambda_2 = 0$$

then

$$H = \frac{z^{1+i\lambda_1}\bar{z}^{1-i\lambda_1}}{(\alpha z + \bar{\alpha}\bar{z} + 1)^{1-\lambda_2+\lambda_1\lambda_3}}, \alpha = \frac{i - \lambda_1}{1 + \lambda_1^2}$$

is a first integral of (21).

Nakayama lemma

In the ring of convergent power series $\mathbb{R}\{\lambda\}$ consider the ideal of functions vanishing along the variety (14), (15). It is obviously generated by

$$a = \lambda_1 + \lambda_3 + \lambda_1\lambda_2, b = \lambda_5, c = \lambda_3\lambda_4$$

and at the second step we shall show that

$$\mathcal{B}_2 =< a, b, c > .$$

We examine first the information obtained from the first and the second Melnikov functions (see the next sections). It follows from (38) that there are elements v_1^2, v_2^2 of the ideal $\mathcal{B}_2$ such that

$$v_1^2(\lambda) = \lambda_1 + \lambda_3 + \dots$$
$$v_2^2(\lambda) = \lambda_5 + \dots$$

where the dots replace some analytic series which vanish of order at least two at $\lambda = 0$.

Nakayama lemma

We can write therefore

$$v_1^2(\lambda) = a + \alpha c + \dots$$
$$v_2^2(\lambda) = b + \beta c + \dots$$

where the dots replace some analytic series which vanish along (14), (15), vanish of order at least two at $\lambda = 0$, and α, β are appropriate constants. We can write finally

$$v_1^2(\lambda) = a\,(1 + O(\lambda)) + b\,O(\lambda) + c\,(\alpha + O(\lambda)) \quad (23)$$
$$v_2^2(\lambda) = a\,O(\lambda) + b\,(1 + O(\lambda)) + c\,(\beta + O(\lambda)). \quad (24)$$

Similarly, the identity (22) implies that under the condition

$$\lambda_1 + \lambda_3 = \lambda_5 = 0$$

there is an element v_3^2 of $\mathcal{B}_2$ such that

$$v_3^2(\lambda) = \lambda_3\lambda_4 + \dots$$

where the dots replace some analytic series vanishing along (14), (15), and vanish of order at least three at $\lambda = 0$.

Nakayama lemma

Without the conditions $\lambda_1 + \lambda_3 = \lambda_5 = 0$, we get

$$v_3^2(\lambda) = \lambda_3\lambda_4 + (\lambda_1 + \lambda_3)(\gamma + O(\lambda)) + \lambda_5(\delta + O(\lambda)) + ...$$

where the dots replace some analytic series which vanish of order at least three at $\lambda = 0$. Thus

$$v_3^2(\lambda) = c(1 + O(\lambda)) + a(\gamma + O(\lambda)) + b(\delta + O(\lambda)).$$

and combining with (23), (24)

$$\begin{pmatrix} v_1^2 \\ v_2^2 \\ v_3^2 \end{pmatrix} = \begin{pmatrix} 1 + O(\lambda) & 0 & \alpha + O(\lambda) \\ 0 & 1 + O(\lambda) & \beta + O(\lambda) \\ \gamma + O(\lambda) & \delta + O(\lambda) & 1 + O(\lambda) \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{25}$$

As the above matrix is invertible for λ close to the origin, then a, b, c belong to $\mathcal{B}_2$, which completes the proof of Theorem 1 .

Poincaré maps

The vector field X_λ, see (9), defines return maps $\mathcal{P}_1, \mathcal{P}_2$ with associated Bautin ideals

$$\mathcal{B}_1 =< v_1^1(\lambda), v_2^1(\lambda), v_3^1(\lambda) >=< \lambda_1, \lambda_3, \lambda_2\lambda_5 > \quad (26)$$

$$\mathcal{B}_2 =< v_1^2(\lambda), v_2^2(\lambda), v_3^2(\lambda) >=< \lambda_1 + \lambda_3 + \lambda_1\lambda_2, \lambda_5, \lambda_3\lambda_4 > . \quad (27)$$

$\mathcal{P}_1, \mathcal{P}_2$ can be divided in the corresponding ideals (26) and (27) as follows, (see Françoise-Yomdin).

The bifurcation functions

$$
\begin{aligned}
\mathcal{P}_1(h;\lambda)(h) - h &= v_1^1(\lambda)(M_1^1(h) + O(\lambda)) + v_2^1(\lambda)(M_2^1(h) + O(\lambda)) \\
&\quad + v_3^1(\lambda)(M_3^1(h) + O(\lambda)) \\
\mathcal{P}_2(h;\lambda)(h) - h &= v_1^2(\lambda)(M_1^2(h) + O(\lambda)) + v_2^2(\lambda)(M_2^2(h) + O(\lambda)) \\
&\quad + v_3^2(\lambda)(M_3^2(h) + O(\lambda)).
\end{aligned}
$$

Definition

The functions (of h)

$$M_1^1(h), M_2^1(h) \text{ and } M_1^2(h), M_2^2(h)$$

are called the first order (or linear) Melnikov functions, associated to the centers at $(0,0)$ and $(0,1)$. The functions

$$M_3^1(h) \text{ and } M_3^2(h)$$

are called the second order (or non-linear) Melnikov functions, associated to the centers at $(0,0)$ and $(0,1)$.

Link with arcs

The terminology is due to J.-P.Françoise-L. Gavrilov-D. Xiao, and it will be justified in what follows. Given an arc,

$$\varepsilon \mapsto \lambda(\varepsilon), \ \varepsilon \in (\mathbb{R}, 0), \ \lambda(0) = 0, \tag{28}$$

we obtain

$$\mathcal{P}_1^1(h; \lambda(\varepsilon))(h) - h = \varepsilon^{k_1}(c_1^1 M_1^1(h) + c_2^1 M_2^1(h) + c_3^1 M_3^1(h) + O(\varepsilon))$$
$$\mathcal{P}_1^2(h; \lambda(\varepsilon))(h) - h = \varepsilon^{k_2}(c_1^2 M_1^2(h) + c_2^2 M_2^2(h) + c_3^2 M_3^2(h) + O(\varepsilon)).$$

Note that not all linear combinations

$$c_1^1 M_1^1(h) + c_2^1 M_2^1(h) + c_3^1 M_3^1(h), c_1^2 M_1^2(h) + c_2^2 M_2^2(h) + c_3^2 M_3^2(h) \tag{29}$$

of Melnikov functions are admissible.

Admissible limit cycles

Definition

Let $K \subset \mathbb{R}^2$ be a compact set. A (i, j) distribution of limit cycles is said to be admissible for X_λ, if for every $\varepsilon > 0$ there exists λ, such that $\|\lambda\| < \varepsilon$ and X_λ has a (i, j) distribution of limit cycles in K.

Let (i, j) be admissible distribution of limit cycles for X_λ in the compact set K. Then there exists a germ of analytic arc (28), such that the one-parameter family of vector fields $X_{\lambda(\varepsilon)}$ allows a distribution (i, j) limit cycles, for ε close to 0. Therefore to compute the possible distributions (i, j) of limit cycles we have to compute the number of zeros i and j of each admissible pair of Melnikov functions (29).

Plan

First Bifurcation functions

In this section we compute, for completeness the first Melnikov functions of X_λ. These results are classical, see Chicone-Shafer, Li-Llibre, Garijo, Gasull, Jarque,... Here we use a simple residue calculus, following Françoise-Yang. If we write X_λ in the form (5)

$$\frac{1}{2}(1-2y)^2 dH + \omega = 0 \tag{30}$$

where

$$H = \frac{x^2+y^2}{2y-1}$$

First Bifurcation functions and punctures on the Riemann surface

Denote

$$\Gamma_h = \{(x,y) \in \mathbb{C}^2 : x^2 + y^2 = (2y-1)h, y \neq \frac{1}{2}\}. \tag{31}$$

which, for $h \neq 0, 1$, is a four-punctured Riemann sphere, where the punctures are at

$$(\pm\frac{\sqrt{-1}}{2}, \frac{1}{2}), \infty^{\pm}. \tag{32}$$

Let

$$\delta(h), \tilde{\delta}(h) \in H_1(\Gamma_h,) \tag{33}$$

be a continuous family of cycles vanishing at the singular points $(0,0)$ and $(0,1)$, when h tends to $h = 0$ or $h = 1$ respectively. These two families of cycles are defined in a neighbourhood of $h = 0$ and $h = 1$ respectively, and hence on the real segment $(0,1)$.

First Bifurcation functions and Riemann sphere

Definition

The first Melnikov functions $M_1(\lambda, h)$, $\tilde{M}_1(\lambda, h)$ associated to the centers $(0,0)$ and $(0,1)$ respectively, are defined by

$$M_1(\lambda, h) = 2\int_{\delta(h)} \omega, \ \tilde{M}_1(\lambda, h) = 2\int_{\tilde{\delta}(h)} \omega.$$

The functions are analytic on $(0,1)$ and therefore can be computed and compared there. This is easy, as they are Abelian integrals on a Riemann sphere, so the computation is reduced to residue calculus.
Following Françoise-Yang, chose an uniformizing variable $z : \bar{\Gamma}_h \to \mathbb{P}^1$ by the formula

$$z = x + i(y-h), i = \sqrt{-1}. \tag{34}$$

If we note $\bar{z} = x - i(y-h)$ (so that $\bar{z}$ is complex conjugate to z when $h \in \mathbb{R}$) we have

$$\Gamma_h = \{(z, \bar{z}) \in \mathbb{C}^2 : z\bar{z} = h(h-1)\}.$$

The images of the four punctures (32) on the curve (31) under $z : \bar{\Gamma}_h \to \mathbb{P}^1$ are

$$z(\infty^+) = \infty, z(\infty^-) = 0, z(\frac{i}{2}, \frac{1}{2}) = -i(h-1), z(-\frac{i}{2}, \frac{1}{2}) = -ih \tag{35}$$

where i is an appropriate determination of $\sqrt{-1}$.

The model of four-punctured Riemann sphere

The model of the four-punctured Riemann sphere Γ_h will be therefore the punctured complex plane $\mathbb{C} \setminus \{a, b, c\}$, where

$$a = -ih, b = -i(h-1), c = 0.$$

$$\alpha, \beta, \gamma, \delta, \tilde{\delta}$$

We have

$$\lim_{h \mapsto 0} a(h) = c, \quad \lim_{h \mapsto 1} b(h) = c$$

and it is easy to check that the vanishing cycles $\delta(h)$ and $\tilde{\delta}(h)$ are represented by "small" simple loops containing a, c for $h \sim 0$, and b, c for $h \sim 1$, as shown on below.

The homology classes

It follows that for the homology classes (denoted by the same letters) holds

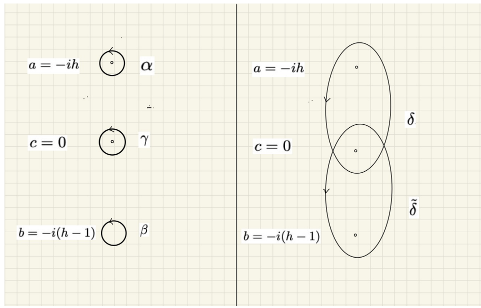

Figure: The loops $\alpha, \beta, \gamma, \delta$ and $\tilde{\delta}$ for $h \in (0, 1)$.

$$\delta = \alpha + \gamma, \; \tilde{\delta} = \beta + \gamma$$

and hence

$$\frac{1}{2} M_1(h) = \int_{\alpha(h)} \omega + \int_{\gamma(h)} \omega, \; \frac{1}{2} \tilde{M}_1(h) = \int_{\beta(h)} \omega + \int_{\gamma(h)} \omega.$$

Simple residue calculus

The explicit computation of M_1 is a simple residue calculus. It was already computed by Françoise and Yang and we reproduce them below. With the notations (3), (11), (11), (11) it follows that

$$M_1(\lambda,h) = -\frac{1}{16}\pi 4h(A_1\frac{4h-1}{8}+A_0), \qquad h<0$$
$$\tilde{M}_1(\lambda,h) = \frac{1}{16}\pi 4(h-1)(B_1\frac{4h-1}{8}+B_0), \qquad h>1$$

where

$$A_1 = 16(\lambda_3+\lambda_1), \qquad A_0 = 2(\lambda_3-3\lambda_1)$$
$$B_1 = A_1, \qquad B_0 = 2(\lambda_3+\lambda_1)-16\lambda_5$$

which implies

$$M_1(\lambda,h) = \lambda_1 M_1^1(h)+\lambda_3 M_2^1(h) = \tag{36}$$
$$\lambda_1[-2\pi h(h-1)]+\lambda_3[-2\pi h^2], \qquad h<0 \tag{37}$$
$$\tilde{M}_1(\lambda,h) = (\lambda_1+\lambda_3)M_1^2(h)+\lambda_5 M_2^2(h) = \tag{38}$$
$$(\lambda_1+\lambda_3)[2\pi h(h-1)]+\lambda_5[-4\pi(h-1)], \qquad h>1. \tag{39}$$

As expected $M_1 \neq \tilde{M}_1$ which allows to construct at a first order all possible distributions (i,j) of limit cycles, such that $i\leq 1, j\leq 1$.

Recall of the first order

Denote

$$\delta(h) = \{(x,y) \in \mathbb{R}^2 : H(x,y) = h\}, h \leq 0$$
$$\tilde{\delta}(h) = \{(x,y) \in \mathbb{R}^2 : H(x,y) = h\}, h \geq 1$$

the family of real ovals of the affine algebraic curve $\Gamma_h \in \mathbb{C}^2$. Using the notations (11), (11), (11) , the following integral formulae for the linear Melnikov functions are well known,
The linear Melnikov functions are given by

$$\frac{1}{2} M_1^1(h) = \int_{\delta(h)} \omega_1, \quad \frac{1}{2} M_2^1(h) = \int_{\delta(h)} \omega_3$$
$$\frac{1}{2} M_1^2(h) = \int_{\tilde{\delta}(h)} \omega_1, \quad \frac{1}{2} M_2^2(h) = \int_{\tilde{\delta}(h)} \omega_5.$$

Proof of the proposition

It is easy to verify that

$$\int_{\delta(h)} \omega_2 = \int_{\delta(h)} \omega_4 = \int_{\delta(h)} \omega_5 = 0$$
$$\int_{\tilde{\delta}(h)} \omega_2 = \int_{\tilde{\delta}(h)} \omega_4 = 0, \int_{\tilde{\delta}(h)} \omega_1 = \int_{\tilde{\delta}(h)} \omega_3.$$

and hence

$$\int_{\delta(h)} \omega = \lambda_1 \int_{\delta(h)} \omega_1 + \lambda_3 \int_{\delta(h)} \omega_3$$
$$\int_{\tilde{\delta}(h)} \omega = (\lambda_1 + \lambda_3) \int_{\tilde{\delta}(h)} \omega_1 + \lambda_5 \int_{\tilde{\delta}(h)} \omega_5.$$

Gelfand-Leray residue

The second order (nonlinear) Melnikov function is given by the second order in λ homogeneous piece of the displacement maps $\mathcal{P}_1 - id, \mathcal{P}_2 - id$. For a differential one-form on $\mathbb{C}^2$ let ω' be the Gelfand-Leray residue of ω with respect to H defined by the identity

$$\omega' \wedge dH = d\omega.$$

The second order Melnikov function of a deformed foliation $dH + \varepsilon\omega = 0$ is defined by the following iterated integral of length two (Gavrilov).

$$\int_{\delta(h)} \omega\omega'$$

with appropriate choice of the path $\delta(h)$. In our case this implies

Second-order Melnikov functions

Proposition

Assume that the linear Melnikov function $M_1^1 = M_2^1 = 0$ *. Then*

$$\frac{1}{4}M_3^1(h) = \int_{\delta(h)} \omega_2\omega_5' + \omega_5\omega_2' \tag{40}$$

Similarly, if $M_1^2 = M_2^2 = 0$*, then*

$$\frac{1}{4}M_3^2(h) = \int_{\tilde{\delta}(h)} (\omega_3 - \omega_1)\omega_5' + \omega_5(\omega_3 - \omega_1)' \tag{41}$$

Proof

The vanishing of $\int_{\delta(h)} \omega$ implies $\lambda_1 = \lambda_3 = 0$. The function $M_3^1(h)$ corresponds to the coefficient $\lambda_2 \lambda_5$ in the iterated integral

$$\int_{\delta(h)} \omega\omega', \omega = \sum_{i=1}^{5} \lambda_i \omega_i.$$

Therefore, assuming in addition that $\lambda_4 = 0$ we get

$$\begin{aligned} \int_{\delta(h)} \omega\omega' &= \int_{\delta(h)} (\lambda_2\omega_2 + \lambda_5\omega_5)(\lambda_2\omega_2 + \lambda_5\omega_5)' \\ &= \lambda_2\lambda_5 \int_{\delta(h)} \omega_2\omega_5' + \omega_5\omega_2' \end{aligned}$$

where we used that $\int_{\delta(h)} \omega_2\omega_2' = \int_{\delta(h)} \omega_5\omega_5' = 0$ (This will be justified latter in the text by using the shuffle formula). The proof of the formula for $M_3^2(h)$ follows the same lines.

Analytic computations

Let

$$H = \frac{x^2 + y^2}{(2y - 1)}$$

be the first integral of X_0, see (1). On each level set

$$\Gamma_h = \{(x, y) \in \mathbb{C}^2 : H(x, y) = h\}$$

holds

$$x^2 + y^2 = h(2y - 1),$$
$$x^2 + (y - h)^2 = h(h - 1).$$

The real level sets $\{(x, y) \in \mathbb{R}^2 : H(x, y) = h\}$, $h \in \mathbb{R}$, are therefore circles centered at $(0, h)$ of radius $R = \sqrt{h(h - 1)}$. The critical values of H are $h = 0$ and $h = 1$. The two period annuli are

$$\{(x, y) \in \mathbb{R}^2 : H(x, y) < 0\}, \ \{(x, y) \in \mathbb{R}^2 : H(x, y) > 1\}.$$

Note that the symmetry $\sigma : y \to 1 - y$ induces $\sigma^*(H) = H - 1$.

Analytic computations

Recall that Γ_h is a four-punctured Riemann sphere, uniformized by the complex parameter

$$z = x + \mathrm{i}(y - h)$$

where

$$x - \mathrm{i}(y - h) = R^2/z.$$

Γ_h is therefore identified with the complex z-plane with three punctures at a, b, c where

$$a = -\mathrm{i}h, b = -\mathrm{i}(h-1), c = 0 \text{ and } R^2 = -ab.$$

In what follows, as in the preceding section, ω is the differential one-form (11), but under the condition that

$$\int_{\delta(h)} \omega = 0$$

or

$$\int_{\tilde{\delta}(h)} \omega = 0.$$

The loops $\delta, \tilde{\delta}$ are represented by circles surrounding a, c or b, c respectively. The one-form ω is holomorphic on Γ_h and has poles at $z = a, b, c, \infty$.
Our purpose is to compute the second Melnikov function of the perturbed equation $dH - \omega = 0$ (the $1/2$ factor of H was skipped for convenience).

Computation of the perturbative part in new coordinates

In the normal form, the perturbative part can be written as:

$$\omega = \lambda_1 \frac{xdy - ydx}{(2y-1)^2} + \lambda_2 \frac{(x^2+y^2)dy}{(2y-1)^2} - \lambda_3 \frac{(x^2+y^2)dx}{(2y-1)^2} + \lambda_4 \frac{(x^2-y^2)dy + 2xydx}{(2y-1)^2} + \lambda_5 \frac{2xydy - (x^2-y^2)dx}{(2y-1)^2}. \tag{42}$$

If we assume that $M_1(h) \equiv 0$, then $\lambda_1 = \lambda_3 = 0$ so

$$\begin{aligned} \omega &= \lambda_2\omega_2 + \lambda_4\omega_4 + \lambda_5\omega_5 \\ &= \lambda_2 \frac{(x^2+y^2)dy}{(2y-1)^2} + \lambda_4 \frac{(x^2-y^2)dy + 2xydx}{(2y-1)^2} \\ &+ \lambda_5 \frac{2xydy - (x^2-y^2)dx}{(2y-1)^2}. \end{aligned}$$

It is easily verified that

$$\mathbb{C}^2 \to \mathbb{C}^2 : (x, y) \mapsto (z, h), \quad h = H(x, y))$$

is a bi-rational transformation of $\mathbb{C}^2$. Therefore we can use z, h coordinates to express $dH - \omega$ and compute the corresponding second Melnikov function.

Expressions in the new coordinates

We have

$$dx = \frac{(z^2 - R^2)}{2z^2}dz + \frac{1}{2}\frac{2h-1}{z}dh$$

$$dy = \frac{1}{2\mathrm{i}z^2}(z^2 + R^2)dz + \frac{[z - \frac{a+b}{2}]}{z}dh.$$

We get by simple substitutions:

$$\frac{x^2 dy}{(2y-1)^2} = \frac{\mathrm{i}}{8}\frac{(z^2-ab)^3}{[z(z-a)(z-b)]^2}dz - \frac{1}{4}\frac{(z^2-ab)^2(z-\frac{a+b}{2})}{z[(z-a)(z-b)]^2}dh.$$

$$\frac{y^2 dy}{(2y-1)^2} = -\frac{\mathrm{i}}{8}\frac{[(z-a)(z-b)+\mathrm{i}z]^2(z^2-ab)}{[z(z-a)(z-b)]^2}dz + \frac{1}{4}\frac{[(z-a)(z-b)+\mathrm{i}z]^2(z-\frac{a+b}{2})}{z[(z-a)(z-b)]^2}dh.$$

Expressions in the new coordinates

$$\frac{2xy}{(2y-1)^2}dx = -\frac{1}{4\mathrm{i}}\frac{(z^2-ab)(z^2+ab)[(z-a)(z-b)+\mathrm{i}z]}{[z(z-a)(z-b)]^2}dz - \frac{(2h-1)}{4\mathrm{i}}\frac{(z^2-ab)[(z-a)(z-b)+\mathrm{i}z]}{z[(z-a)(z-b)]^2}dh.$$

$$\frac{2xy}{(2y-1)^2}dy = \frac{1}{4}\frac{(z^2-ab)^2[(z-a)(z-b)+\mathrm{i}z]}{[z(z-a)(z-b)]^2}dz - \frac{1}{2\mathrm{i}}\frac{(z^2-ab)(z-\frac{a+b}{2})[(z-a)(z-b)+\mathrm{i}z]}{z[(z-a)(z-b)]^2}dh.$$

$$\frac{-x^2+y^2}{(2y-1)^2}dx = \frac{1}{8}\frac{((z^2-ab)^2+[(z-a)(z-b)+\mathrm{i}z]^2)(z^2+ab)}{[z(z-a)(z-b)]^2}dz + \frac{2h-1}{8}\frac{(z^2-ab)^2+[(z-a)(z-b)+\mathrm{i}z]^2}{z[(z-a)(z-b)]^2}dh.$$

The second-order Melnikov function defined by an iterated integral

From $\omega = Fdz + \Phi dH$, we get:

$$d\omega = (F'_H - \Phi'_z)dH \wedge dz.$$

The Gelfand-Leray derivative of ω is defined (modulo dH) by

$$\omega' = (F'_H - \Phi'_z)dz.$$

The associated second-order Melnikov function is defined as the iterated integral (of length two):

$$M_2(h) = -\int \omega\omega'. \tag{43}$$

From previous calculation the only terms which contribute effectively are:

$$M_2(h) = -(\int \omega_2\omega'_5 + \int \omega_5\omega'_2)\lambda_2\lambda_5. \tag{44}$$

The main result we show here is that such an iterative integral can be computed by residues. For this purpose, we have first to compute ω_2, ω_5 and ω'_2, ω'_5 in the coordinates (z, h) and to determine their partial fraction decompositions.

Shuffle formula

We recall an important formula (particular case of the shuffle formula for any couple of one-forms ω_0, ω_1:

$$\int \omega_0\omega_1 + \int \omega_1\omega_0 = \int \omega_0 . \int \omega_1 . \tag{45}$$

In particular this yields that if $\int \omega_0 = 0$ or $\int \omega_1 = 0$, then

$$\int \omega_0\omega_1 = -\int \omega_1\omega_0 . \tag{46}$$

Computation of ω_2 and its derivatives

We note that:

$$\omega_2 = \frac{1}{2} h d(\ln(2y-1)) = \frac{hdy}{2y-1} = \frac{x^2+y^2}{(2y-1)^2} dy, \tag{47}$$

and thus we get:

$$\omega_2' = \frac{dy}{2y-1} = \frac{1}{2\mathrm{i} z^2(2y-1)} (z^2 + R^2) dz + \frac{[z - \frac{a+b}{2}]}{z(2y-1)} dh. \tag{48}$$

Computation of ω_2 and its derivatives

If we change coordinates (x, y) into (z, h), we obtain:

$$\omega_2 = h[\frac{z^2 - ab}{2z(z-a)(z-b)}dz + \mathrm{i}\frac{(z - \frac{a+b}{2})}{(z-a)(z-b)}dh], \tag{49}$$

and thus:

$$\omega_2 = F_2(z,h)dz + \phi_2(z,h)dh, \tag{50}$$

with

$$\begin{aligned} F_2(z,h) &= \tfrac{h}{2}[-\tfrac{1}{z} + \tfrac{1}{z-a} + \tfrac{1}{z-b}] \\ \Phi_2(z,h) &= \tfrac{\mathrm{i}h}{2}[\tfrac{1}{z-a} + \tfrac{1}{z-b}]. \end{aligned} \tag{51}$$

Computation of ω_2 and its derivatives

We see that:

$$\int \omega_2 = \int_{H=h} F_2 dz = 0. \tag{52}$$

The shuffle formula implies for instance:

$$\int \omega_2\omega_2 = -\int \omega_2\omega_2 = 0. \tag{53}$$

Similar (but more complicated) computation can be done for ω_5 and its derivatives

Computation of $-\int \omega_5\omega_2'$

We begin by the observation that:

$$\omega_2' = \frac{1}{2}[-\frac{1}{z} + \frac{1}{z-a} + \frac{1}{z-b}]dz + ...(dH), \tag{54}$$

and so:

$$\int_{H=h} \omega_2' = 0, \tag{55}$$

hence we can apply the shuffle formula and obtain:

$$-\int \omega_5\omega_2' = \int \omega_2'\omega_5. \tag{56}$$

This displays:

$$\begin{aligned}&\int \omega_2'\omega_5 = \int \tfrac{1}{2}[-\tfrac{1}{z} + \tfrac{1}{z-a} + \tfrac{1}{z-b}]dz\omega_5 = \\ &-\tfrac{\mathrm{i}(h-1)}{2}\int[-\tfrac{1}{z} + \tfrac{1}{z-a} + \tfrac{1}{z-b}]dz[\tfrac{1}{z} - \tfrac{1}{z-a} + \tfrac{1}{z-b}]dz \\ &+\tfrac{1}{2}\int_{h=h}[-\tfrac{1}{z} + \tfrac{1}{z-a} + \tfrac{1}{z-b}]\{\tfrac{1}{2}z + h(h-1)[\tfrac{1}{z} + \tfrac{1}{z-a}] + \tfrac{(h-1)^2}{2}\tfrac{1}{z-b}\}dz.\end{aligned} \tag{57}$$

Computation of $-\int \omega_5 \omega_2'$

Note that the second expression can be readily computed by residue. The first component breaks into four pieces that we compute by the shuffle formula:

$$\int [-\frac{1}{z} + \frac{1}{z-a}]dz[\frac{1}{z} - \frac{1}{z-a}] = 0, \tag{58}$$

$$\int [\frac{1}{z-b}]dz[\frac{1}{z} - \frac{1}{z-a}]dz = \int [-\frac{1}{z} + \frac{1}{z-a}]dz[\frac{1}{z-b}]dz, \tag{59}$$

$$\int [\frac{1}{z-b}]dz[\frac{1}{z-b}]dz = 0. \tag{60}$$

This gives the contribution:

$$\begin{aligned} &-\mathrm{i}(h-1)[\int_{H=h}(-\tfrac{1}{z} + \tfrac{1}{z-a})\mathrm{Log}(z-b)dz = \\ &-\mathrm{i}(h-1)(2\pi\mathrm{i})[-\mathrm{Log}(-b) + \mathrm{Log}(a-b)] = \\ &2\pi(h-1)[-\mathrm{Log}(-\mathrm{i}(1-h)) + \mathrm{Log}(-\mathrm{i})] = -2\pi(h-1)\mathrm{Log}(1-h). \end{aligned} \tag{61}$$

Computation of $-\int \omega_2 \omega_5'$

The last component contributes to the sum of residues:

$$\frac{\pi}{2}h - \frac{\pi h(h-1)}{2} + \frac{\pi h}{2} - \frac{\pi}{2}(h-1)^2\left(\frac{1}{h-1} + 1\right), \tag{62}$$

and all together this holds:

$$-\int \omega_5 \omega_2' = \pi(2h - h^2) - 2\pi(h-1)\mathrm{Log}(1-h). \tag{63}$$

Similar computation of $-\int \omega_2 \omega_5'$ can be done.

Final expression of $M_3^1(\lambda, h)$

To conclude we have proved the:

Theorem

The value of $M_3^1(\lambda, h)$ is:

$$\begin{aligned} M_3^1(\lambda, h) &= [\pi(2h - h^2) - 2\pi(h-1)\mathrm{Log}(1-h) + 2\pi(h)\mathrm{Log}(1-h) + 2\pi h^2]\lambda_2\lambda_5 \\ &= 2\pi[h + \frac{h^2}{2} + \mathrm{Log}(1-h)]\lambda_2\lambda_5. \end{aligned}$$

Similar computation can be done for

$$M_3^2(\lambda, h) = [P(h) + \mathrm{Log}(h)]\lambda_3\lambda_4 = [O(1-h)^3]\lambda_3\lambda_4.$$

In the article, this is checked independently by a monodromy analysis of the iterated integrals.

Plan

Blow up of an ideal

Let $\mathbb{C}\{\lambda\}$ be the ring of convergent power series at $\lambda = 0$, where $(\lambda_1, ...\lambda_n) \in \mathbb{C}^n$, and

$$\mathcal{B} = (v_1, \ldots, v_N) \subset \mathbb{C}\{\lambda\}$$

be an ideal with zero set

$$Z(\mathcal{B}) = \{\lambda \in (\mathbb{C}^n, 0) : v_1(\lambda) = v_2(\lambda) = \cdots = v_N(\lambda) = 0\}$$

The blowup $\Gamma_{\mathcal{B}} \subset (\mathbb{C}^n, 0) \times \mathbb{P}^{N-1}$ of $(\mathbb{C}^n, 0)$ with center $\mathcal{B}$ is the analytic closure of the graph of the map

$$C^n \setminus Z(\mathcal{B}) \to \mathbb{P}^{N-1}$$
$$\lambda \mapsto [v_1(\lambda) : \cdots : v_N(\lambda)]$$

with projection on the first factor

$$\pi_{\mathcal{B}} : \Gamma_{\mathcal{B}} \subset (\mathbb{C}^n, 0) \times \mathbb{P}^{N-1} \to (\mathbb{C}^n, 0).$$

Here $[v_1(\lambda) : \cdots : v_N(\lambda)]$ is the projectivization of $(v_1(\lambda), \ldots, v_N(\lambda))$. The exceptional divisor

$$E_{\mathcal{B}} = \pi^{-1}(0) \subset \mathbb{P}^{N-1}$$

is therefore a well defined closed algebraic set. The importance of $E_{\mathcal{B}}$ lies in the fact that it is in bijective correspondence with the projectivized set of bifurcation (or Melnikov) functions, computed in the preceding sections, see J.-P.Françoise-L. Gavrilov-D. Xiao.

Blow-up of a product of ideals

Suppose that $\mathcal{B}_1, \mathcal{B}_2 \subset \mathbb{C}\{\lambda\}$ be two ideals

$$\mathcal{B}_1 = (v_1^1, \dots, v_{N_1}^1)$$
$$\mathcal{B}_2 = (v_1^2, \dots, v_{N_1}^2)$$

and consider the direct product

$$\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \subset \mathbb{C}\{\lambda\} \times \mathbb{C}\{\lambda\}.$$

We note that $\mathcal{B}$ is also an ideal and consider the corresponding blowup

$$\Gamma_{\mathcal{B}} \subset (\mathbb{C}^n, 0) \times \mathbb{P}^{N_1-1} \times \mathbb{P}^{N_2-1}$$

defined as the analytic closure of the graph of the map

$$\begin{aligned} C^n \setminus Z(\mathcal{B}) &\to \mathbb{P}^{N_1-1} \times \mathbb{P}^{N_2-1} \\ \lambda &\mapsto ([v_1^1(\lambda) : \dots : v_{N_1}^1(\lambda)], [v_1^1(\lambda) : \dots : v_{N_2}^1(\lambda)]) \end{aligned}$$

with corresponding exceptional divisor

$$E_{\mathcal{B}_1 \times \mathcal{B}_2} = \pi^{-1}(0) \subset \mathbb{P}^{N_1-1} \times \mathbb{P}^{N_2-1}.$$

Exceptional divisor $E_{\mathcal{B}_1\times\mathcal{B}_2}$

To the end of the present section we compute $E_{\mathcal{B}_1\times\mathcal{B}_2}$ in the case when

$$\mathcal{B}_1 =< v_1^1(\lambda), v_2^1(\lambda), v_3^1(\lambda) >=< \lambda_1, \lambda_3, \lambda_2\lambda_5 >$$
$$\mathcal{B}_2 =< v_1^2(\lambda), v_2^2(\lambda), v_3^2(\lambda) >=< \lambda_1+\lambda_3+\lambda_1\lambda_2, \lambda_5, \lambda_3\lambda_4 > .$$

It follows with same proof as FGX that

Proposition

The projectivized set of pairs of Melnikov functions computed in the preceding section are in bijective correspondence with the points on the exceptional divisor $E_{\mathcal{B}_1\times\mathcal{B}_2}$.

Irreducible components of $E_{\mathcal{B}_1 \times \mathcal{B}_2}$

The main result of the present section is

Theorem

The exceptional divisor

$$E_{\mathcal{B}_1 \times \mathcal{B}_2} \subset \mathbb{P}^2 \times \mathbb{P}^2$$

has three irreducible components as follows

$$\{([c_1^1 : c_2^1 : c_3^1], [c_1^2 : c_2^2 : c_3^2]) : c_1^2 = c_3^2 = 0\} \tag{64}$$

$$\{([c_1^1 : c_2^1 : c_3^1], [c_1^2 : c_2^2 : c_3^2]) : c_1^1 + c_2^1 = 0, c_3^1 = 0\} \tag{65}$$

$$\{([c_1^1 : c_2^1 : c_3^1], [c_1^2 : c_2^2 : c_3^2]) : c_3^1 = c_3^2 = 0\}. \tag{66}$$

Irreducible components of $E_{\mathcal{B}_1\times\mathcal{B}_2}$

A point $(P_1, P_2) \in (\mathbb{P}^2, \mathbb{P}^2)$ belongs to $E_{\mathcal{B}_1\times\mathcal{B}_2}$ if and only if there is an arc

$$\varepsilon \mapsto \lambda(\varepsilon) = (\lambda_1(\varepsilon), \dots, \lambda_6(\varepsilon)), \lambda(0) = 0 \tag{67}$$

such that the vector

$$([v_1^1(\lambda(\varepsilon)) : v_2^1(\lambda(\varepsilon)) : v_3^1(\lambda(\varepsilon))], [v_1^2(\lambda(\varepsilon)) : v_2^2(\lambda(\varepsilon)) : v_3^2(\lambda(\varepsilon))])$$

tends to the vector (P_1, P_2) as ε tends to 0. It is easy to show now that the components (64),(65), (66) belong to $E_{\mathcal{B}_1\times\mathcal{B}_2}$.

Irreducible components of $E_{\mathcal{B}_1 \times \mathcal{B}_2}$

For instance, for (65) we may consider the family of arcs

$$\varepsilon \mapsto \lambda(\varepsilon) = (\varepsilon, \varepsilon^2, -\varepsilon + \lambda_3^0 \varepsilon^2, -\lambda_4^0 \varepsilon, \lambda_5^0 \varepsilon^2)$$

and then

$$\lim_{\varepsilon \mapsto 0} ([v_1^1(\lambda(\varepsilon)) : v_2^1(\lambda(\varepsilon)) : v_3^1(\lambda(\varepsilon))], [v_1^2(\lambda(\varepsilon)) : v_2^2(\lambda(\varepsilon)) : v_3^2(\lambda(\varepsilon))])$$
$$= ([1 : -1 : 0], [\lambda_3^0 : \lambda_5^0 : \lambda_4^0])$$

The other inclusion are also obvious.

Irreducible components of $E_{\mathcal{B}_1 \times \mathcal{B}_2}$

Next, we consider an arbitrary arc (28) and we must show that

$$\lim_{\varepsilon \mapsto 0}([v_1^1(\lambda(\varepsilon)) : v_2^1(\lambda(\varepsilon)) : v_3^1(\lambda(\varepsilon))], [v_1^2(\lambda(\varepsilon)) : v_2^2(\lambda(\varepsilon)) : v_3^2(\lambda(\varepsilon))])$$

belongs to one of (64),(65), (66). For this purpose we note that for fixed λ_2, λ_4, the generators v_i^j of $\mathcal{B}_1, \mathcal{B}_2$ are linear homogeneous in $\lambda_1, \lambda_3, \lambda_5$.

Irreducible components of $E_{\mathcal{B}_1\times\mathcal{B}_2}$

Therefore we shall consider separately each of the cases

$$d_1=\min_{i=1,3,5} d_i,\ d_3=\min_{i=1,3,5} d_i,\ d_5=\min_{i=1,3,5} d_i.$$

where

$$\lambda_1=O(\varepsilon^{d_1}),\lambda_3=O(\varepsilon^{d_3}),\lambda_5=O(\varepsilon^{d_5}).$$

- The case $d_1=\min_{i=1,3,5} d_i$ We put

  $$\lambda_1=\lambda_1^0\varepsilon^{d_1}+\dots,\lambda_3=\lambda_3^0\varepsilon^{d_3}+\dots,\lambda_5=\lambda_5^0\varepsilon^{d_5}+\dots$$

  and hence

  $$\lim_{\varepsilon\mapsto 0}[\lambda_1:\lambda_3:\lambda_2\lambda_5]=[\lambda_1^0:\lambda_3^0:0].$$

  If $\lambda_1+\lambda_3=O(\varepsilon^{d_1})$ then

  $$\lim_{\varepsilon\mapsto 0}[\lambda_1+\lambda_3+\lambda_1\lambda_2:\lambda_5:\lambda_3\lambda_4]=[*,*,0]$$

  and therefore the limit is in the set (66). If, however $\lambda_1+\lambda_3=O(\varepsilon^{\tilde{d}_1})$ where $\tilde{d}_1>d_1$, then

  $$\lim_{\varepsilon\mapsto 0}[\lambda_1:\lambda_3:\lambda_2\lambda_5]=[1:-1:0].$$

  and the limit is in the set (65).

Irreducible components of $E_{\mathcal{B}_1 \times \mathcal{B}_2}$

- The case $d_3 = \min_{i=1,3,5} d_i$ We may suppose in addition that $d_3 < d_1$ (otherwise we are in the preceding case). Then we check immediately that the limit is in the set (66)
- The case $d_5 = \min_{i=1,3,5} d_i$ We may suppose in addition that $d_5 < d_1$ and $d_5 < d_3$ (otherwise we are in one of the preceding two cases). Therefore

$$\lim_{\varepsilon \mapsto 0} [\lambda_1 + \lambda_3 + \lambda_1 \lambda_2 : \lambda_5 : \lambda_3 \lambda_4] = [0 : 1 : 0]$$

  and we are in the case (64).

This completes the proof of Theorem 6.

Distributions of limit cycles

We determine the possible distributions (i, j) of limit cycles of small quadratic deformations (5) of the quadratic vector field (1) on the finite plane $\mathbb{R}^2$. This excludes the limit cycles, which bifurcate from "infinity".

Definition

We say that the germ of a family of vector fields $X_{a,b}$

$$X_{a,b} : \begin{cases} \dot{x} = -y - x^2 + y^2 + \sum_{0 \le i,j \le 2} a_{ij} x^i y^j, \\ \dot{y} = \quad x - 2xy - \sum_{0 \le i,j \le 2} b_{ij} x^i y^j \end{cases}$$

has an admissible distribution (i, j) of limit cycles, if there is a sequence $(a_k, b_k)_k$ in the parameter space $\{(a, b)\}$ such that for every sufficiently big $R \in \mathbb{R}$ the following holds true : every vector field X_{a_k, b_k} has exactly i limit cycles surrounding the equilibrium point near $(0, 0)$, exactly j limit cycles surrounding the equilibrium point near $(0, 1)$, and these limit cycles are contained in the disc $\{(x, y) \in \mathbb{R}^2 : \|(x, y\| < R\}$.

Cyclicity

The maximal value of i is therefore the cyclicity $Cycl(\Pi_1, X_{a,b})$ of the open period annulus containing $(0,0)$, the maximal value of j is the cyclicity $Cycl(\Pi_2, X_{a,b})$ of the open period annulus containing $(0,1)$, and finally

$$\max_{i,j} i + j = Cycl(\mathbb{R}^2, X_{a,b})$$

Recall that the cyclicity $Cycl(\Pi, X_{a,b})$ of an open set $\Pi \subset \mathbb{R}^2$ with respect to the germ of a family of vector fields $X_{a,b}$ is, roughly speaking, the maximal number of limit cycles which bifurcate from an arbitrary compact set $K \subset \Pi$ when $a, b \sim 0$.

Admissible Distributions

The main result of the paper is

Theorem

The distribution (i, j) of limit cycles is admissible if and only if $i + j \leq 2$.

Without loss of generality we replace the germ of families $X_{a,b}$ by X_λ, see (9). The first return maps $\mathcal{P}_1, \mathcal{P}_2$ parameterized by the restriction $h = H(x, y)$ of the first integral on a cross-section to the annulus Π_1 or Π_2 can be divided in the corresponding ideals (26) and (27) as follows

$$\begin{aligned}\mathcal{P}_1(h;\lambda)(h) - h = {} & v_1^1(\lambda)(M_1^1(h) + O(\lambda)) + v_2^1(\lambda)(M_2^1(h) + O(\lambda)) \\ & + v_3^1(\lambda)(M_3^1(h) + O(\lambda)) \\ \mathcal{P}_2(h;\lambda)(h) - h = {} & v_1^2(\lambda)(M_1^2(h) + O(\lambda)) + v_2^2(\lambda)(M_2^2(h) + O(\lambda)) \\ & + v_3^2(\lambda)(M_3^2(h) + O(\lambda))\end{aligned}$$

where the Melnikov functions M_i^j were computed in the preceding sections.

Associated arcs

It follows, that if (i, j) is an admissible distribution of limit cycles for X_λ, then there exists a germ of analytic arc

$$\varepsilon \mapsto \lambda(\varepsilon),\ \varepsilon \in (\mathbb{R}, 0),\ \lambda(0) = 0$$

such that the one-parameter family of vector fields $X_{\lambda(\varepsilon)}$ allows a distribution (i, j) of limit cycles, for ε close to 0. For such an arc we obtain

$$\mathcal{P}_1(h; \lambda(\varepsilon))(h) - h = \varepsilon^{k_1}(c_1^1 M_1^1(h) + c_2^1 M_2^1(h) + c_3^1 M_3^1(h) + O(\varepsilon))$$
$$\mathcal{P}_2(h; \lambda(\varepsilon))(h) - h = \varepsilon^{k_2}(c_1^2 M_1^2(h) + c_2^2 M_2^2(h) + c_3^2 M_3^2(h) + O(\varepsilon))$$

Therefore to compute the distribution (i, j) of limit cycles we have to compute the number of zeros i and j of each admissible pair of bifurcation functions

$$c_1^1 M_1^1(h) + c_2^1 M_2^1(h) + c_3^1 M_3^1(h), c_1^2 M_1^2(h) + c_2^2 M_2^2(h) + c_3^2 M_3^2(h)$$

Limit cycles and bifurcation functions

According to section 6 and (36), (38), the bifurcation function associated to the first period annulus is co-linear to

$$h[c_1^1(h-1)+c_2^1h]+c_3^1M_3^1(h)$$

and the bifurcation function associated to the second annulus is

$$(h-1)[c_1^2h-2c_2^2]+c_3^2M_3^2(h).$$

According to FGX, the admissible pairs of vectors

$$[c_1^1:c_2^1:c_3^1],[c_1^2:c_2^2:c_3^2]\in\mathbb{P}^2$$

are in one-to-one correspondance to the points on the exceptional divisor

$$E_{\mathcal{B}_1\times\mathcal{B}_2}\subset\mathbb{P}^2\times\mathbb{P}^2$$

described in Theorem 6. We consider each of the three irreducible components of $E_{\mathcal{B}_1\times\mathcal{B}_2}$ separately.

Irreducible components of $E_{\mathcal{B}_1\times\mathcal{B}_2}$ and limit cycles

In the component (64) we have $c_1^2 = c_3^2 = 0$ so the bifurcation function associated to the second annulus Π_2 is co-linear to $h-1$. Thus no limit cycles bifurcate from Π_2 and from Π_1, the number of limit cycles is given by the number of zeros of $c_1^1 h^2 + c_1^2 h + c_3^1 Log(1-h)$ which is at most 2.
In the component (65) we have $c_1^1 + c_2^1 = 0, c_3^1 = 0$ and hence the bifurcation function associated to the first period annulus Π_1 is co-linear to h. Thus no limit cycles bifurcate from Π_1 and at most two limit cycles bifurcate from Π_2.
In the component (66) we have $c_3^1 = c_3^2 = 0$ and hence the bifurcation functions associated to the period annuli are co-linear to

$$h[c_1^1(h-1) + c_2^1 h],\ (h-1)[c_1^2(h-1) - 2c_2^2].$$

Therefore in each period annulus at most one limit cycle can bifurcate. This completes the proof.

Plan

Conclusions and Perspectives

The main novelties of this article are:

- The explicit computation of the pair of Bautin ideals
- The self-intersection of the Lotka-Volterra center set at the double quadratic Lotka-Volterra system (see the conference of Lubomir Gavrilov at GADEPS)
- The proof that we can stop at order two to compute the number of limit cycles born at infinite order of the perturbation.
- Two approaches (analytical and geometrical) to find the second-order Bifurcation functions.
- Provide a new application of the use of the Nash space of arcs and of the main theorem of FGX.
- Note that there are two different arguments used to obtain the maximal number of limit cycles. One is directly computed from the Bautin ideals and the determination of the singular fiber of the blow-up of the product of the two Bautin ideals (use of the FGX theorem). The other uses the bound of the number of zeros of the functions of h, $M_j^i(h), j = 1, 2, 3; i = 1, 2)$.

Although we recall that the methods we have used do not allow to keep track of all the limit cycles which are born at the boundaries of the period annuli. This issue has been adressed in several other bifurcation settings ("alien cycles"). This is certainly an interesting perspective for further researches.
Another important perspective would be to try to extend the outline of a general bifurcation theory of plane systems of infinite co-dimension that we have introduced here, in particular to other reversible quadratic double centers.

Revisiting Petrowski-Landis

Petrowski-Landis article certainly contains interesting ideas that have not been yet explored. Roughly speaking they consider the viewpoint of complex foliations and they propose to introduce a notion of "regularly situated complex limit cycles". The article is organized in three parts. The first one is devoted to the definition of these special complex limit cycles and their main properties. It should be certainly clarified. The second part consists in a careful study of a perturbative situation. Amazingly, this perturbative situation is exactly the one we have studied in our article! It should be certainly revisited...