Geometry of certain foliations in the complex projective plane

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Foliations of degree d on $\mathbb{P}^2_{\mathbb{C}}$

In homogeneous coordinates $[x, y, z]$ they are given by a homogeneous polynomial vector field of degree d:

$$X = A(x,y,z)\partial_x + B(x,y,z)\partial_y + C(x,y,z)\partial_z,$$

with $\gcd(A, B, C) = 1$. If $\mathcal{R} = x\partial_x + y\partial_y + z\partial_z$, the 2-dimensional distribution $\langle X, \mathcal{R}\rangle$ on $\mathbb{C}^3$ induces a line distribution on $\mathbb{P}^2_{\mathbb{C}}$ whose integral curves are the leaves of the foliation $\mathcal{F}$ defined by X.
Dually, $\mathcal{F}$ is defined by $\ker \Omega = \langle X, \mathcal{R}\rangle$, where

$$\Omega = \imath_X \imath_{\mathcal{R}}(dx \wedge dy \wedge dz) = \begin{vmatrix} dx & dy & dz \\ x & y & z \\ A & B & C \end{vmatrix} = Pdx + Qdy + Rdz.$$

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Notice that P, Q, R are homogeneous polynomials of degree $d + 1$.
Pulling-back Ω to the affine chart $\iota : \mathbb{C}^2 \hookrightarrow \mathbb{P}^2_{\mathbb{C}}$, $\iota(x, y) = [x, y, 1]$, we obtain $\omega = \iota^*\Omega = P(x, y, 1)dx + Q(x, y, 1)dy$, which has degree $\leq d$ if and only if $P(x, y, 0) = Q(x, y, 0) = 0$, i.e. if the line at infinity $z = 0$ associated to the affine chart ι is invariant by $\mathcal{F}$.

The space $\mathbb{F}(d)$ of foliations of degree d

In the affine chart (x, y) the foliation $\mathcal{F} \in \mathbb{F}(d)$ is given by

$$\omega = \sum_{0 \le i,j \le d} p_{ij} x^i y^j dx + \sum_{0 \le i,j \le d} q_{ij} x^i y^j dy + \sum_{i+j=d} r_{ij} x^i y^j (x dy - y dx)$$

up to multiplication by a non-zero scalar, i.e. $\mathbb{F}(d)$ is an open dense subset of $\mathbb{P}_{\mathbb{C}}^{N_d}$, $N_d = 2\frac{(d+1)(d+2)}{2} + (d+1) - 1 = d^2 + 4d + 2$.

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The group $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}}) = \mathrm{PGL}(3, \mathbb{C})$ acts naturally on $\mathbb{F}(d)$ by means of $(g, \mathcal{F}) \mapsto g^*\mathcal{F}$. If $\mathcal{F} \in \mathbb{F}(d)$ is defined by $\ker\Omega$ and $g = [g_{ij}] \in \mathrm{PGL}(3, \mathbb{C})$ then $g^*\mathcal{F}$ is defined by the kernel of

$$g^*\Omega = \Omega\Big|_{\substack{x = g_{11}x + g_{12}y + g_{13}z \\ y = g_{21}x + g_{22}y + g_{23}z \\ z = g_{31}x + g_{32}y + g_{33}z}}$$

We define the orbit and the isotropy subgroup of $\mathcal{F} \in \mathbb{F}(d)$ by

$$\mathcal{O}(\mathcal{F}) = \{g^*\mathcal{F} \,|\, g \in \mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})\},\ \mathrm{Aut}(\mathcal{F}) = \{g \in \mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}}) \,|\, g^*\mathcal{F} = \mathcal{F}\}.$$

We have that $\dim \mathcal{O}(\mathcal{F}) + \dim \mathrm{Aut}(\mathcal{F}) = \dim \mathrm{PGL}(3, \mathbb{C}) = 8$.

Small closed orbits

Theorem [Cerveau, Deserti, Garba-Belko, Meziani, 2010]:
If $d \geq 2$ and $\mathcal{F} \in \mathbb{F}(d)$ then $\dim \mathrm{Aut}(\mathcal{F}) \leq 2$. If in addition $\dim \mathrm{Aut}(\mathcal{F}) = 2$ then the Lie algebra of $\mathrm{Aut}(\mathcal{F})$ is not abelian.

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Theorem A: If $d \geq 2$ and $\mathcal{F} \in \mathbb{F}(d)$ with $\dim \mathrm{Aut}(\mathcal{F}) = 2$ then $\mathcal{F}$ is conjugated either to $\mathcal{F}_1^d = [dx + y^d dy]$ or $\mathcal{F}_2^d = [y^d dx + dy]$. Moreover the orbits $\mathcal{O}(\mathcal{F}_1^d)$ and $\mathcal{O}(\mathcal{F}_2^d)$ are closed and different.

Generalizing the cases $d = 2$ by [C,D,GB,M, 2010] and $d = 3$ by [Alcántara, Ronzón-Lavie, 2016].

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Idea of the proof: Classify all the affine 2-dimensional Lie subalgebras $\mathfrak{a}$ of $\mathfrak{X}(\mathbb{P}^2_{\mathbb{C}}) \simeq \mathfrak{sl}(3,\mathbb{C}) = \{M \in \mathrm{M}_{3\times 3}(\mathbb{C}) \mid \mathrm{Tr}(M) = 0\}$ up to conjugation and impose that $(L_A\Omega) \wedge \Omega = 0$ for each $A \in \mathfrak{a}$.
In coordinates $[x, y, z]$ the isomorphism $\mathfrak{sl}(3,\mathbb{C}) \xrightarrow{\sim} \mathfrak{X}(\mathbb{P}^2_{\mathbb{C}})$ writes as

$$M \mapsto A = (\partial_x, \partial_y, \partial_z) M \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

In the affine chart $z = 1$ we replace ∂_z by $-x\partial_x - y\partial_y$.

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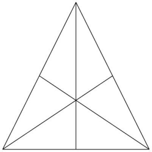
Remark: Loud's isochronous center $[(x - \frac{x^2}{2} + \frac{y^2}{2})dx - y(x-1)dy]$ is conjugated (via $\mathrm{PGL}(3,\mathbb{C})$) to the degree $d = 2$ Fermat foliation $\mathcal{F}_F^d = [x^d\partial_x + y^d\partial_y + z^d\partial_z] = [(y^d - y)dx - (x^d - x)dy] \in \mathbb{F}(d)$ having $3d$ different (complex) invariant lines:

$$xyz(x^{d-1} - y^{d-1})(x^{d-1} - z^{d-1})(y^{d-1} - z^{d-1}) = 0.$$



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Indeed, if $g(x, y) = (1 + x - y, i(1 - x - y)) \in \mathrm{PGL}(3, \mathbb{C})$ then

$$\frac{1}{2} g^* \left(y(1-x)dy + \left(x - \frac{x^2}{2} + \frac{y^2}{2} \right) dx \right) = (y^2 - y)dx - (x^2 - x)dy.$$

The orbit $\mathcal{O}(\mathcal{F}_F^d)$ has dimension 8 and its closure contains $\mathcal{O}(\mathcal{F}_2^d)$.

Inflection divisor and convex foliations

Definition [Pereira, 2001]: The inflection divisor of $\mathcal{F} \in \mathbb{F}(d)$ defined by $X = A\partial_x + B\partial_y + C\partial_z$ is the degree $3d$ algebraic curve

$$I_{\mathcal{F}}(x,y,z) = \begin{vmatrix} x & y & z \\ X(x) & X(y) & X(z) \\ X^2(x) & X^2(y) & X^2(z) \end{vmatrix} = 0,$$

consisting in the inflection points of the leaves of $\mathcal{F}$, including all its invariant lines. The foliation $\mathcal{F}$ is convex when $I_{\mathcal{F}}$ is entirely composed by invariant lines.

Example: $\mathcal{F}_F^d = [x^d\partial_x + y^d\partial_y + z^d\partial_z]$ is convex but $\mathcal{F}_1^d$ is not.

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Theorem [Favre-Pereira, 2015, after Schlomiuk, Vulpe, 2004]:
$\mathbb{F}_C(2) = \overline{\mathcal{O}(\mathcal{F}_F^2) \cup \mathcal{O}([x^2\partial_x + y^2\partial_y]) \cup \mathcal{O}}(\mathcal{F}_2^2)$. Moreover,
$\mathbb{F}_C(2) = \overline{\mathcal{O}((x^2-x)\partial_x + (y^2-y)\partial_y)} \supset \overline{\mathcal{O}(x^2\partial_x + y^2\partial_y)} \supset \mathcal{O}(\mathcal{F}_2^2)$.

In fact, as $t \to \infty$,

$$\frac{1}{t^{d+1}}(tx,ty)^*((x^d-x)dy-(y^d-y)dx) = (x^d - \frac{x}{t^{d-1}})dy - (y^d - \frac{y}{t^{d-1}})dx \to x^d dy - y^d dx.$$

Basins of attraction

Definition: The basin of attraction of $\mathcal{F} \in \mathbb{F}(d)$ is

$$\mathbb{B}(\mathcal{F}) = \{\mathcal{G} \in \mathbb{F}(d) \,|\, \mathcal{F} \in \overline{\mathcal{O}(\mathcal{G})}\} = \{\mathcal{G} \in \mathbb{F}(d) \,|\, \mathcal{O}(\mathcal{F}) \subset \overline{\mathcal{O}(\mathcal{G})}\}.$$

Remark: If $\mathcal{F} \notin \mathbb{F}_C(d)$ and $\mathcal{G} \in \mathbb{B}(\mathcal{F})$ then $\mathcal{G} \notin \mathbb{F}_C(d)$.

Theorem [C,D,G-B,M, 2010]: $\mathbb{B}(\mathcal{F}_1^2) = \mathbb{F}(2) \setminus \mathbb{F}_C(2)$ is open dense.

This means that for every degree 2 foliation $\mathcal{F}$ which is not conjugated to $(x^2 - x)\partial_x + (y^2 - y)\partial_x$, nor $x^2\partial_x + y^2\partial_y$, nor $\partial_x + y^2\partial_y$, there exists $g \in \mathrm{PGL}(3,\mathbb{C})$ such that $g^*\mathcal{F}$ is arbitrarily close to $y^2\partial_x + \partial_y$ in $\mathbb{F}(2)$.

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Recall that $\dim \mathbb{F}(d) = d^2 + 4d + 2$. Assume $d \geq 2$.

Theorem B: $\dim \mathbb{B}(\mathcal{F}_1^d) \geq \dim \mathbb{F}(d) - (d - 3)$ if $d \geq 3$.
In particular, $\mathbb{B}(\mathcal{F}_1^3)$ is open dense in $\mathbb{F}(3)$.

Theorem C: $\dim \mathbb{B}(\mathcal{F}_2^d) \geq \dim \mathbb{F}(d) - (d - 1)$.

Theorem D: $\dim(\mathbb{B}(\mathcal{F}_1^d) \cap \mathbb{B}(\mathcal{F}_2^d)) \geq \dim \mathbb{F}(d) - 3d$.

Degeneracy and non-degeneracy criteria

Definition: A foliation $\mathcal{F}$ degenerates onto $\mathcal{G}$ if $\mathcal{G} \in \overline{\mathcal{O}(\mathcal{F})} \setminus \mathcal{O}(\mathcal{F})$.
If $\mathbb{C} \ni t \mapsto g_t \in \mathrm{PGL}(3, \mathbb{C})$ is continuous and $\mathcal{G} = \lim_{t \to \infty} g_t^* \mathcal{F}$ is not conjugated to $\mathcal{F}$ then $\mathcal{F}$ degenerates onto $\mathcal{G}$ (denoted by $\mathcal{F} \to \mathcal{G}$).

Remark: If $\mathcal{F} \to \mathcal{G}$ then $\dim \mathcal{O}(\mathcal{F}) > \dim \mathcal{O}(\mathcal{G})$ and $\deg I_{\mathcal{F}}^{\mathrm{inv}} \leq \deg I_{\mathcal{G}}^{\mathrm{inv}}$.

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If $C : f(x, y) = 0$ is a non-invariant curve and $p = (x_0, y_0) \in C$ the tangency order $\mathrm{Tang}(\mathcal{F}, C, p) = \dim_{\mathbb{C}} \mathbb{C}\{x - x_0, y - y_0\}/(f, X(f))$, where X is a local vector field defining $\mathcal{F}$ near p.

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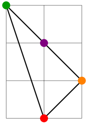
Proposition 1: (a) If $\mathcal{F} \to \mathcal{F}_1^d$ then $\deg I_{\mathcal{F}}^{\mathrm{tr}} \geq d - 1$.
(b) If there is p regular with $\mathrm{Tang}(\mathcal{F}, T_p^{\mathbb{P}} \mathcal{F}, p) = d$ then $\mathcal{F} \to \mathcal{F}_1^d$.

Proposition 2: (a) If $\mathcal{F} \to \mathcal{F}_2^d$ then $\mathcal{F}$ possesses a singularity s whose linear part has equal non-zero eigenvalues ($\mathrm{BB}(\mathcal{F}, s) = 4$).
(b) If $\mathcal{F}$ possesses a singularity s with equal non-zero eigenvalues and a non-invariant line $L \ni s$ with $\mathrm{Tang}(\mathcal{F}, L, s) = d$ then $\mathcal{F} \to \mathcal{F}_2^d$.

Quasi-homogeneous degeneracies via Newton's polygon

If $\omega = \sum_{(i,j)\in I_x} a_{ij} x^{i-1} y^j dx + \sum_{(i,j)\in I_y} b_{ij} x^i y^{j-1} dy$ with $a_{ij}, b_{ij} \neq 0$, the Newton's polygon $N(\omega)$ of ω is the convex hull of $I_x \cup I_y \subset \mathbb{R}^2$.

Example: If $\omega = (1 + y^2)dx + (x^2 + y^2)dy$ then $N(\omega) =$ 

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If $L = \{ai + bj = c\} \subset \partial N(\omega)$ and $g_t(x,y) = (t^a x, t^b y)$ then

$$\lim_{t^{\pm 1}\to\infty} t^{-c} g_t^* \omega = \sum_{(i,j)\in I_x\cap L} a_{ij}x^{i-1}y^j dx + \sum_{(i,j)\in I_y\cap L} b_{ij}x^i y^{j-1} dy$$

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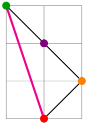
Example: If $\omega = (1+y^2)dx + (x^2+y^2)dy$ then $N(\omega) =$ 

- $g_t(x,y) = (t^3x, ty) \Rightarrow t^{-3}g_t^*\omega = dx + y^2dy + t^2y^2dx + t^4x^2dy$ tends to $dx + y^2dy$ as $t \to 0$, which is invariant by $3x\partial_x + y\partial_y$.

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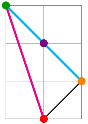
Example: If $\omega = (1 + y^2)dx + (x^2 + y^2)dy$ then $N(\omega) =$ 

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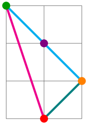
Example: If $\omega = (1 + y^2)dx + (x^2 + y^2)dy$ then $N(\omega) =$ 

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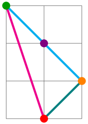
- $g_t(x,y) = (tx, ty) \Rightarrow t^{-3}g_t^*\omega = y^2dx + (x^2 + y^2)dy + t^{-2}dx$ tends to $y^2dx + (x^2 + y^2)dy$ as $t \to \infty$, invariant by $x\partial_x + y\partial_y$.
- $g_t(x,y) = (tx, y/t) \Rightarrow t^{-1}g_t^*\omega = dx + x^2dy + t^{-2}y^2dx + t^{-3}y^2dy$ tends to $dx + x^2dy$ as $t \to \infty$, invariant by $x\partial_x - y\partial_y$.

Quasi-homogeneous degeneracies via Newton's polygon

If $\omega = \sum_{(i,j)\in I_x} a_{ij} x^{i-1} y^j dx + \sum_{(i,j)\in I_y} b_{ij} x^i y^{j-1} dy$ with $a_{ij}, b_{ij} \neq 0$, the Newton's polygon $N(\omega)$ of ω is the convex hull of $I_x \cup I_y \subset \mathbb{R}^2$. If $L = \{ai + bj = c\} \subset \partial N(\omega)$ and $g_t(x,y) = (t^a x, t^b y)$ then

$$\lim_{t^{\pm 1}\to\infty} t^{-c} g_t^* \omega = \sum_{(i,j)\in I_x \cap L} a_{ij} x^{i-1} y^j dx + \sum_{(i,j)\in I_y\cap L} b_{ij} x^i y^{j-1} dy$$

which is invariant by $ax\partial_x + by\partial_y$.

Example: If $\omega = (1+y^2)dx + (x^2+y^2)dy$ then $N(\omega) =$ 

- $g_t(x,y) = (t^3 x, ty) \Rightarrow t^{-3} g_t^*\omega = dx + y^2 dy + t^2 y^2 dx + t^4 x^2 dy$ tends to $dx + y^2 dy$ as $t \to 0$, which is invariant by $3x\partial_x + y\partial_y$.
- $g_t(x,y) = (tx, ty) \Rightarrow t^{-3} g_t^*\omega = y^2 dx + (x^2+y^2)dy + t^{-2} dx$ tends to $y^2 dx + (x^2+y^2)dy$ as $t \to \infty$, invariant by $x\partial_x + y\partial_y$.
- $g_t(x,y) = (tx, y/t) \Rightarrow t^{-1} g_t^*\omega = dx + x^2 dy + t^{-2} y^2 dx + t^{-3} y^2 dy$ tends to $dx + x^2 dy$ as $t \to \infty$, invariant by $x\partial_x - y\partial_y$.

$$\overline{\mathcal{O}(\mathcal{F})} \supset \overline{\mathcal{O}(\mathcal{H})} \supset \mathcal{O}(\mathcal{F}_1^2) \cup \mathcal{O}(\mathcal{F}_2^2).$$

Degeneracy onto $\mathcal{F}_1^d$

Proof of Proposition 1: (a) follows from $I_{\mathcal{F}_1^d} = y^{d-1}z^{2d+1}$ noting that $y=0$ is not invariant by $\mathcal{F}_1^d = [dx + y^d dy] = [y^d\partial_x - \partial_y]$.

(b) Fix affine coordinates (x,y) with $p=(0,0)$ and $T_p^{\mathbb{P}}\mathcal{F} = \{x=0\}$. Then $\mathcal{F}$ is defined by $\omega = (1+a(x,y))dx + (c(y)+xb(x,y))dy$ and $X = (c(y)+xb(x,y))\partial_x - (1+a(x,y))\partial_y$ with $a(0,0)=0$ and $c(0)=0$. Since the ideal $(x, X(x)) = (x, c(y))$ and

$$\mathrm{Tang}(\mathcal{F}, T_p^{\mathbb{P}}\mathcal{F}, p) = \dim_{\mathbb{C}} \mathbb{C}\{x,y\}/(x,c(y)) = d$$

we deduce that $c(y) = cy^d$ with $c \neq 0$. Taking the family of automorphisms $g_t(x,y) = (\frac{cx}{t^{d+1}}, \frac{y}{t}) \in \mathrm{PGL}(3,\mathbb{C})$ we obtain that

$$\frac{t^{d+1}}{c} g_t^*\omega = dx + y^d dy + \left[a\left(\frac{cx}{t^{d+1}}, \frac{y}{t}\right)dx + \frac{x}{t} b\left(\frac{cx}{t^{d+1}}, \frac{y}{t}\right)dy\right]$$

tends to $dx + y^d dy$ as $t \to \infty$.

Theorem B: Basin of attraction of $\mathcal{F}_1^d$, $d \geq 3$

It can be checked that the set $\Sigma \subset \mathbb{F}(d) \times \mathbb{P}^2_{\mathbb{C}}$ consisting in $(\mathcal{F}, p)$ such that $p \notin \mathrm{Sing}(\mathcal{F})$ and $\mathrm{Tang}(\mathcal{F}, T_p^{\mathbb{P}}\mathcal{F}, p) = d$ is defined by

$$\begin{pmatrix} X(x) \\ X(y) \end{pmatrix}(p) \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{vmatrix} X(x) & X^j(x) \\ X(y) & X^j(y) \end{vmatrix}(p) = 0, \; j = 2, \ldots, d,$$

where X is a polynomial vector field defining $\mathcal{F}$ in an affine chart (x, y) containing the point p. Hence $\dim \Sigma \geq \dim \mathbb{F}(d) + 2 - (d-1)$.

By Chevalley's theorem the projection $\pi_1(\Sigma)$ of Σ into $\mathbb{F}(d)$ is a constructible set (it contains an open dense subset of its closure).

The set $U \subset \mathbb{F}(d)$ consisting in foliations with totally transverse and reduced inflection divisor is open dense and contains Jouanolou's foliation $\mathcal{J}^d = [y^d \partial_x + z^d \partial_y + x^d \partial_z] \in \pi_1(\Sigma)$.

If $\pi_2 : \mathbb{F}(d) \times \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ and $\mathcal{F} \in U$ then $\pi_2(\pi_1^{-1}(\mathcal{F}) \cap \Sigma)$ is finite:

If $I_X = \begin{vmatrix} X(x) & X^2(x) \\ X(y) & X^2(y) \end{vmatrix} = KL$ and $X(I_X) = \begin{vmatrix} X(x) & X^3(x) \\ X(y) & X^3(y) \end{vmatrix} = X(K)L + KX(L) = KL'$ then $\gcd(K, L) = 1$, $X(K) = KL''$ and $\{K = 0\} \subset I_{\mathcal{F}}$ is invariant!

Theorem B: Basin of attraction of $\mathcal{F}_1^d$, $d \geq 3$

It can be checked that the set $\Sigma \subset \mathbb{F}(d) \times \mathbb{P}^2_{\mathbb{C}}$ consisting in $(\mathcal{F}, p)$ such that $p \notin \mathrm{Sing}(\mathcal{F})$ and $\mathrm{Tang}(\mathcal{F}, T_p^{\mathbb{P}}\mathcal{F}, p) = d$ is defined by

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Hence $\mathcal{J}^d \in \Sigma_1 := \pi_1(\Sigma) \cap U \subset \mathbb{B}(\mathcal{F}_1^d)$ and

$$\dim \Sigma_1 = \dim \Sigma \geq \dim \mathbb{F}(d) - (d-3).$$

Idea of the proof of Theorem C

If $s = (0,0) \in \mathrm{Sing}(\mathcal{F})$ has equal non-zero eigenvalues ($\mathrm{BB}(\mathcal{F}, s) = 4$) and $L = \{x = 0\}$ satisfies $\mathrm{Tang}(\mathcal{F}, L, s) = d$ then $\mathcal{F}$ is defined by

$$\omega = (xdy - ydx) + y^d dy + a(x,y)dx + xb(x,y)dy,$$

with $a(0,0) = b(0,0) = 0$ and

$$\lim_{t \to 0} \frac{1}{t^{d+1}}\left((t^d x, ty)^* \omega\right) \to xdy - ydx + y^d dy \in \mathcal{O}(\mathcal{F}_2^d).$$

$\Sigma_2 = \{\mathcal{F} \in \mathbb{F}(d) \,|\, \exists L \ni s,\ \mathrm{BB}(\mathcal{F}, s) = 4,\ \mathrm{Tang}(\mathcal{F}, L, s) = d\}$ has codimension $\leq d - 1$ and it is contained in $\mathbb{B}(\mathcal{F}_2^d)$.

Idea of the proof of Theorem C and Theorem D

If $s=(0,0)\in\mathrm{Sing}(\mathcal{F})$ has equal non-zero eigenvalues ($\mathrm{BB}(\mathcal{F},s)=4$) and $L=\{x=0\}$ satisfies $\mathrm{Tang}(\mathcal{F},L,s)=d$ then $\mathcal{F}$ is defined by

$$\omega=(xdy-ydx)+y^d dy+a(x,y)dx+xb(x,y)dy,$$

with $a(0,0)=b(0,0)=0$ and

$$\lim_{t\to 0}\frac{1}{t^{d+1}}\left((t^d x,ty)^*\omega\right)\to xdy-ydx+y^d dy\in\mathcal{O}(\mathcal{F}_2^d).$$

$\Sigma_2=\{\mathcal{F}\in\mathbb{F}(d)\,|\,\exists L\ni s,\ \mathrm{BB}(\mathcal{F},s)=4,\ \mathrm{Tang}(\mathcal{F},L,s)=d\}$ has codimension $\leq d-1$ and it is contained in $\mathbb{B}(\mathcal{F}_2^d)$.

The foliation $\mathcal{H}^d=[(x^d+y^d)dx+x^d dy]\in\Sigma_1\cap\Sigma_2$ so that $\overline{\mathcal{O}(\mathcal{H}^d)}\supset\mathcal{O}(\mathcal{F}_1^d)\cup\mathcal{O}(\mathcal{F}_2^d)$, hence $\mathbb{B}(\mathcal{H}^d)\subset\mathbb{B}(\mathcal{F}_1^d)\cap\mathbb{B}(\mathcal{F}_2^d)$.
The set of foliations defined by

$$(x^d+y^d+A_{d-1}(x,y))dx+(x^d+B_{d-1}(x,y))dy\to\mathcal{H}^d$$

in some affine chart (x,y) of $\mathbb{P}^2_{\mathbb{C}}$ has codimension

$$(d^2+4d+2)-(2(1+\cdots+d-1)+2)=3d\quad\text{in }\mathbb{F}(d).$$

More details are avalaible in the preprint:

S. Bedrouni, D. Marín, Geometry of certain foliations on the complex projective plane, arXiv:2101.11509v4, to appear in The Annali della Scuola Normale Superiore di Pisa.

Thanks for your attention!