Relay systems and some bifurcations from infinity

(joint work by Emilio Freire, E. Ponce, Javier Ros & Elisabet Vela)

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Physica D: Nonlinear Phenomena
Volume 444, February 2023, 133586

Hopf bifurcation at infinity in 3D Relay systems

E. Freire, E. Ponce, J. Ros, E. Vela

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Abstract

A complete analysis of the limit cycle bifurcation from infinity in 3D Relay systems, which belong to the class of three-dimensional symmetric discontinuous piecewise linear systems with two zones, is presented.

A criticality parameter is found, whose sign determines the character of the bifurcation. When such non-degeneracy parameter vanishes, a higher co-dimension bifurcation takes place, giving rise to the emergence of a curve of saddle–node bifurcations of periodic orbits, which allows to determine parameter regions where two limit cycles coexist.

The theoretical results are applied to a specific family of 3D relay systems, where several high co-dimension bifurcation points are detected, organizing the bifurcation set of the family.

A previous work: Hopf Bifurcation from infinity in planar discontinuous systems

J. Math. Anal. Appl. 496 (2021) 124818

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

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Limit cycles from a monodromic infinity in planar piecewise linear systems

Emilio Freire [a], Enrique Ponce [a], Joan Torregrosa [b,c,*], Francisco Torres [a]

[a] *Departamento Matemática Aplicada II, E.T.S. Ingeniería, Universidad de Sevilla, Camino de los Descubrimientos, 41092 Sevilla, Spain*
[b] *Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Cerdanyola del Vallès, Barcelona, Spain*
[c] *Centre de Recerca Matemàtica, Campus de Bellaterra, 08193 Cerdanyola del Vallès, Barcelona, Spain*

Hopf Bifurcation from infinity in 3D relay systems

- The analysis of bifurcations from infinity cannot be forgotten in order to get a complete overview of the dynamical behaviour to be found in a given dynamical system.
- Within the realm of piecewise linear systems, such analysis requires some adaptation of the usual techniques. Starting from the closing equations method for the analysis of periodic orbits, suitable reciprocal coordinates for their intersection points with the discontinuity manifold are introduced.

Hopf Bifurcation from infinity in 3D relay systems

- In the specific case of 3D relay systems, we will show how the analysis provides useful information about the limit cycle bifurcation set, allowing to detect some parameter regions where one or more limit cycles appear.
- This work is the extension to the discontinuous vector fields realm of a similar analysis in symmetric piecewise linear systems with three zones:

Hopf bifurcation at infinity in 3D symmetric piecewise linear systems. Application to a Bonhoeffer–van der Pol oscillator

E. Freire, E. Ponce, J. Ros, E. Vela, A. Amador. Nonlinear Analysis: Real World Applications 54 (2020) 103112

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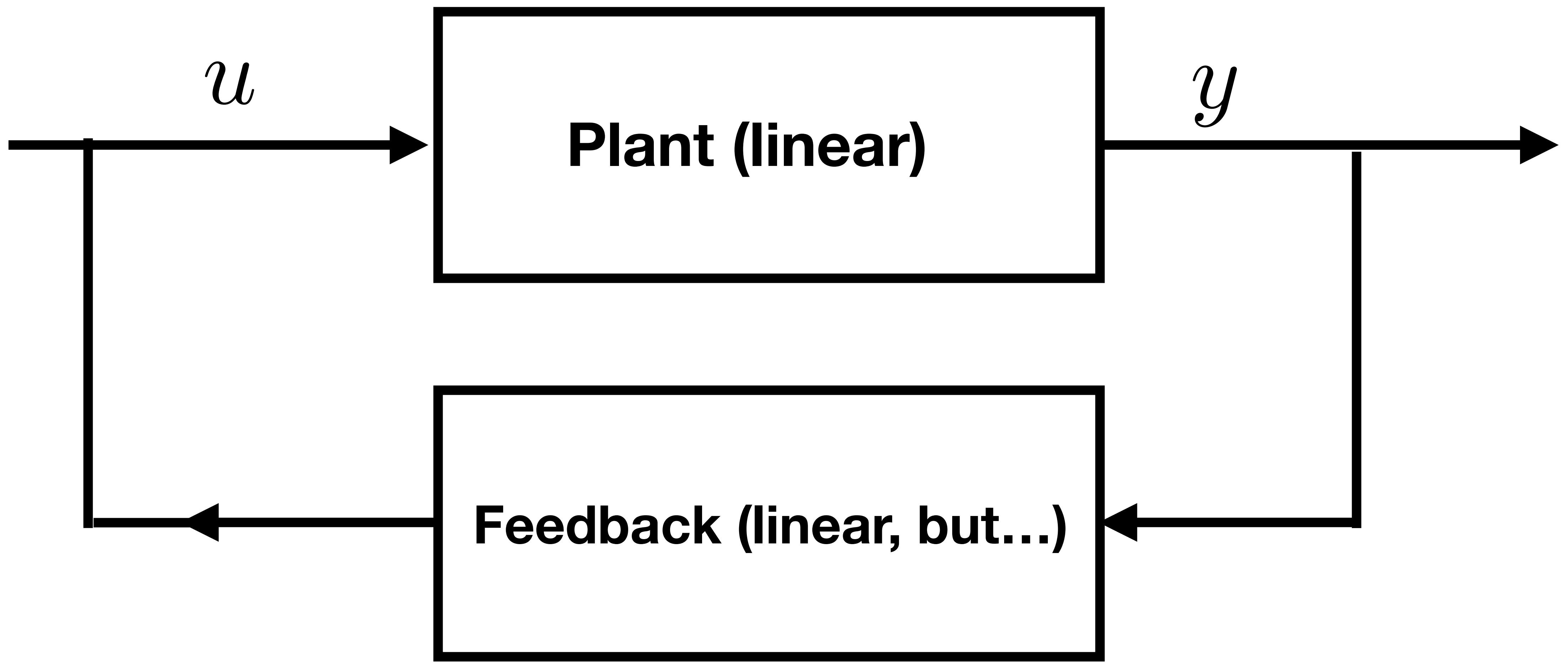
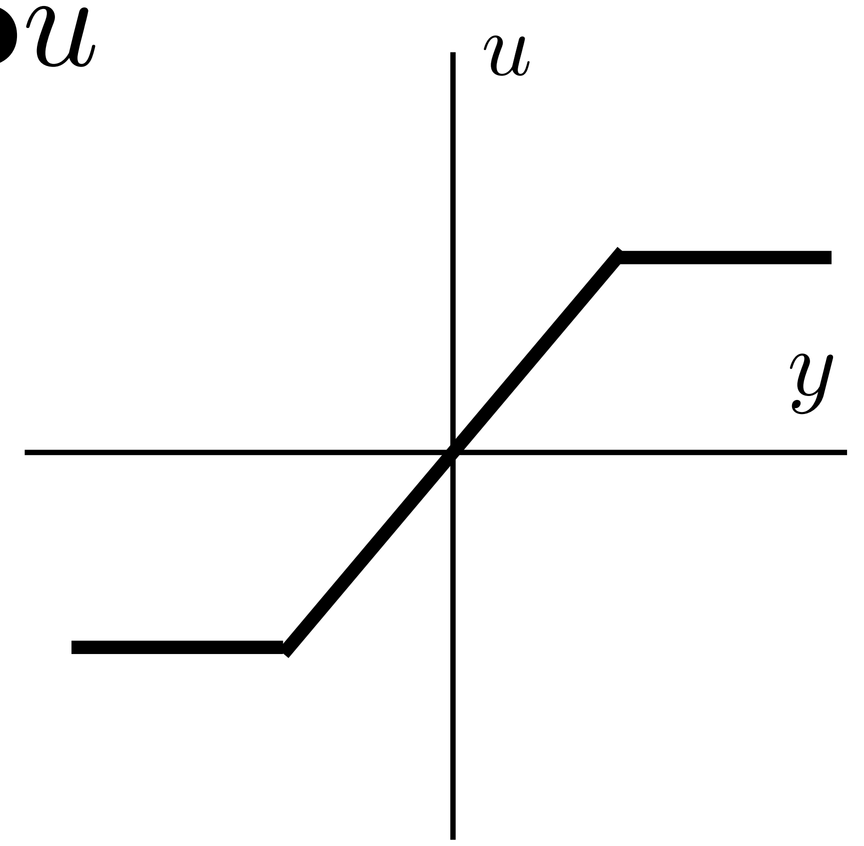
Piecewise linear systems coming from control systems



state equations of the plant → $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}u$

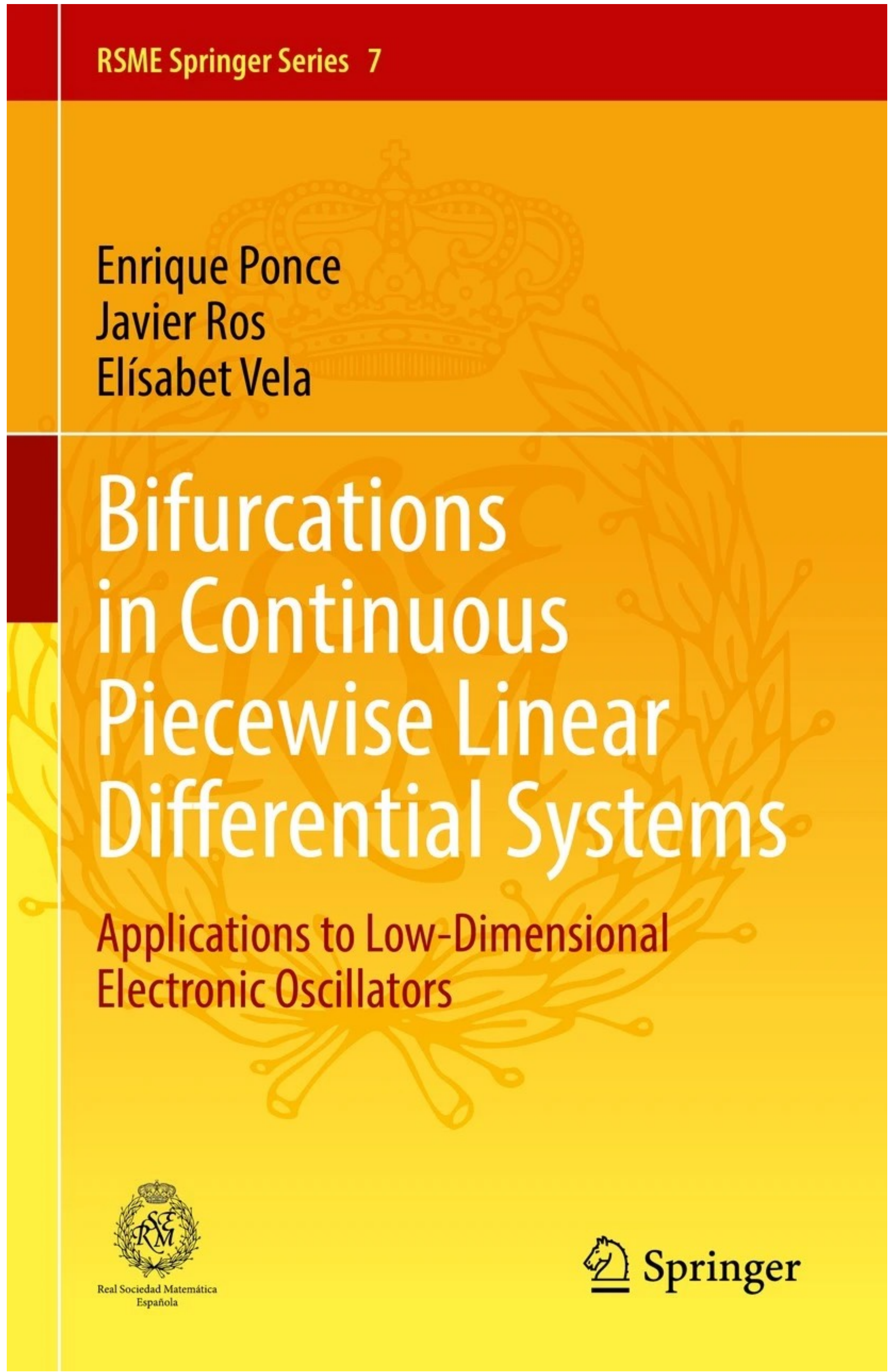
output from states → $y = \mathbf{c}^\top \mathbf{x}$

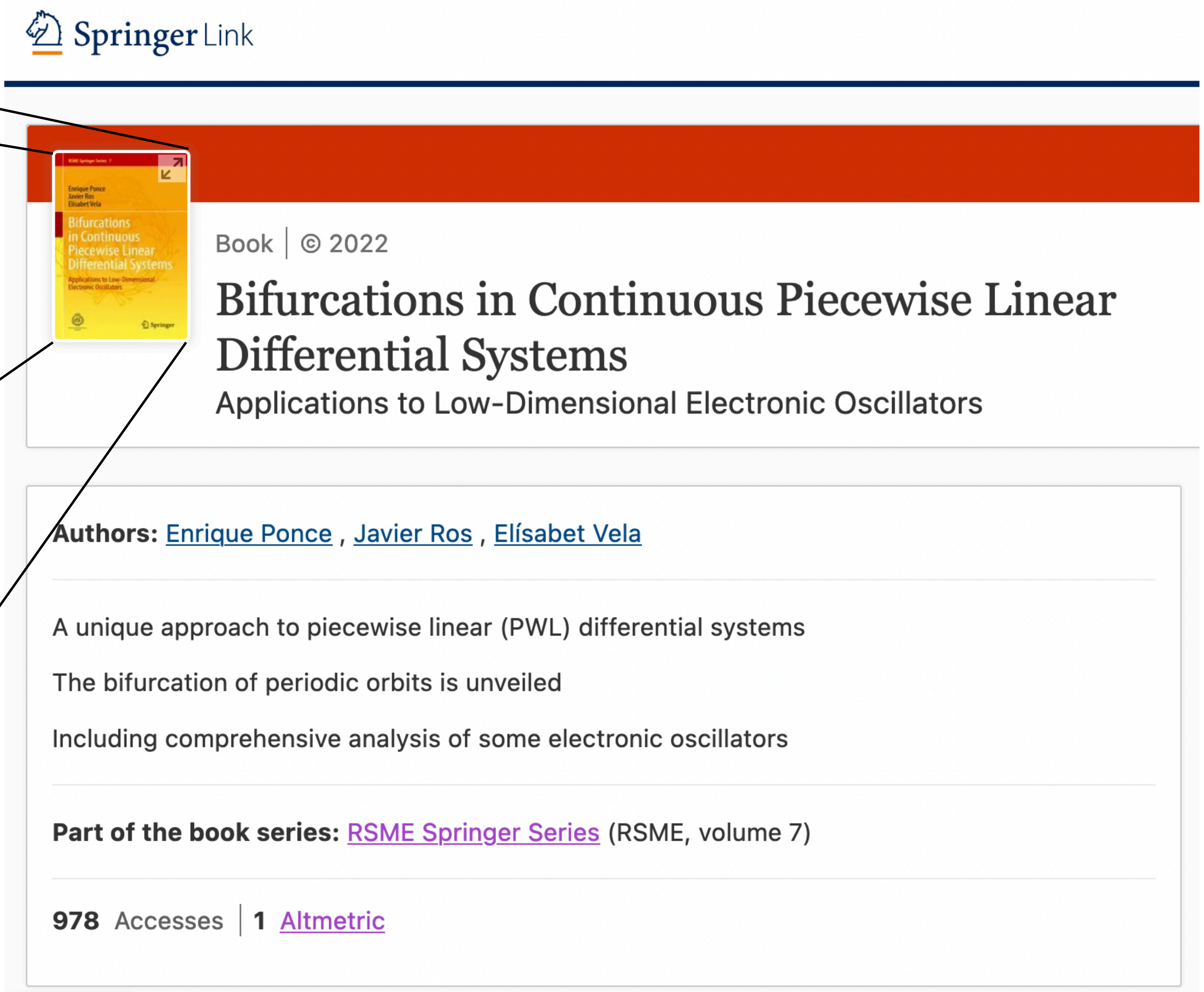
input is a feedback of the output → $u = \mathrm{sat}(y)$



$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}\,\mathrm{sat}(\mathbf{c}^\top \mathbf{x})$$

For continuous piecewise linear systems,





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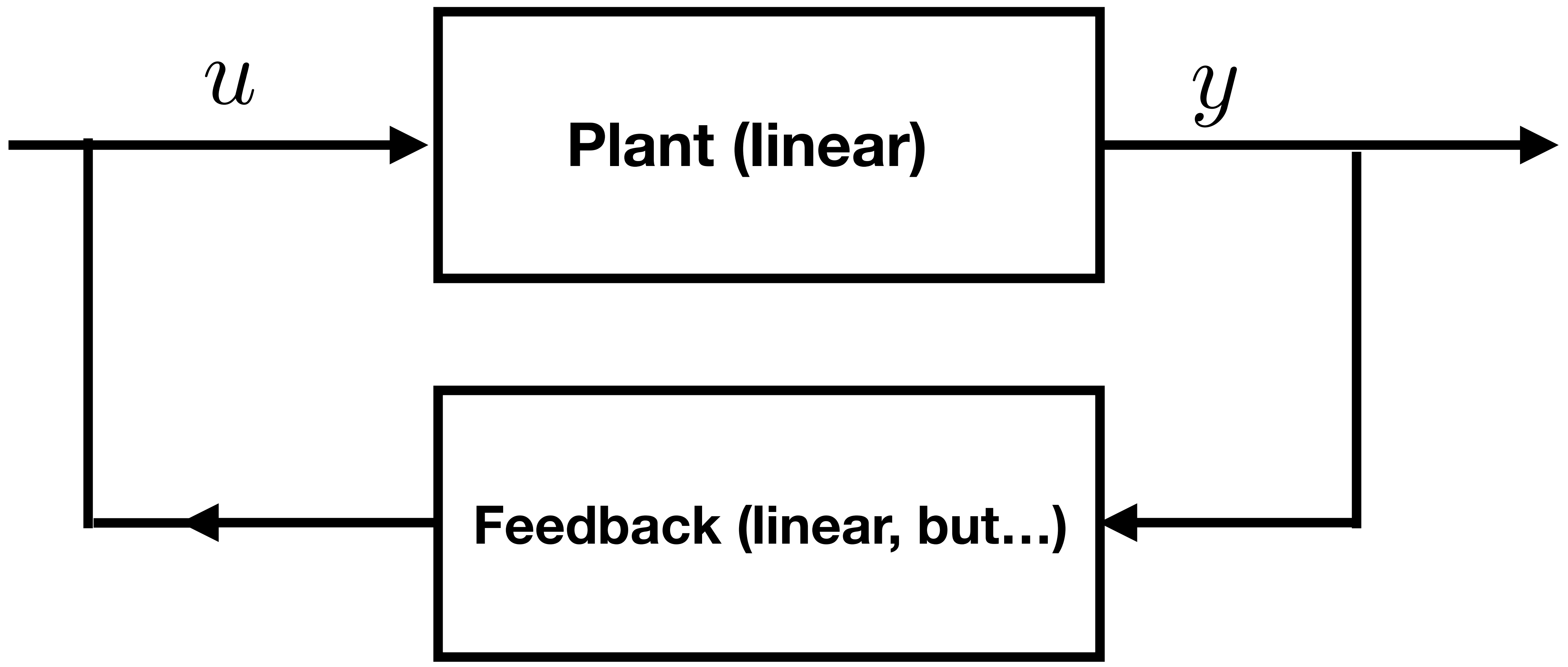
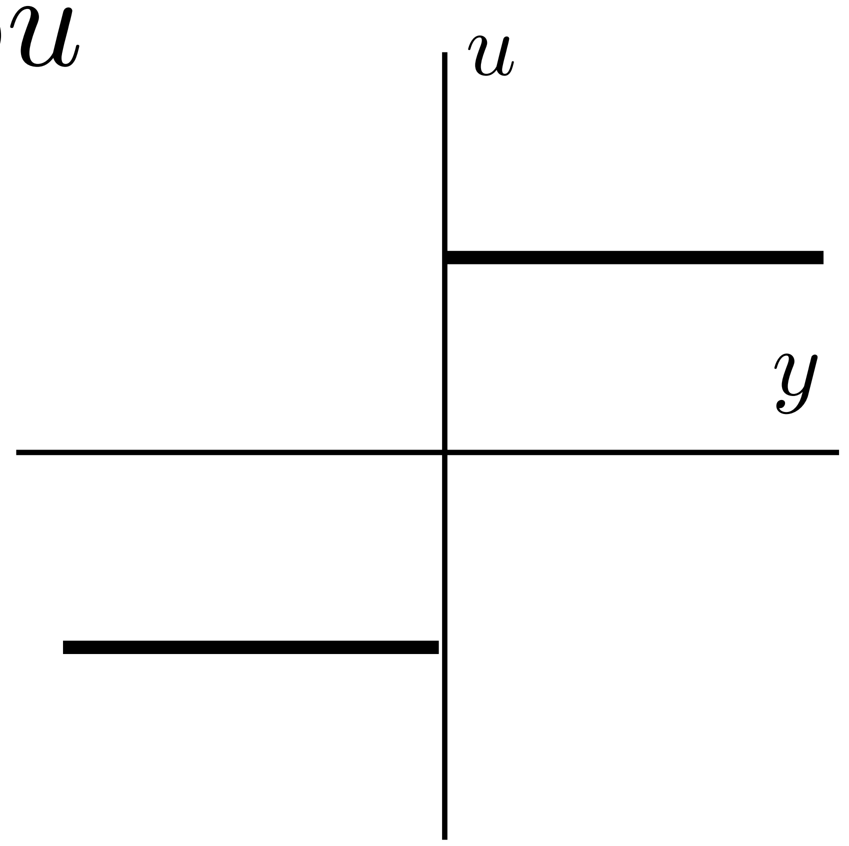
Canonical form for 3D relay systems



state equations of the plant → $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}u$

output from states → $y = \mathbf{c}^\top \mathbf{x}$

input is a feedback of the output → $u = \operatorname{sign}(\mathbf{x})$



$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}\,\operatorname{sign}(\mathbf{c}^\top \mathbf{x})$$

Canonical form for 3D relay systems

For $\mathbf{x} = (x, y, z)^\top \in \mathbb{R}^3$, we consider symmetric discontinuous 3D relay systems

$$\dot{\mathbf{x}} = A\mathbf{x} - \mathbf{b}\varphi\left(\mathbf{c}^\top \mathbf{x}\right)$$

where φ is the sign function. Here, the dot represents derivative with respect to the time τ, the vector $\mathbf{b} = (b_1, b_2, b_3)^\top \in \mathbb{R}^3$ is constant, $\mathbf{c} = \mathbf{e}_1$ (the first canonical vector of $\mathbb{R}^3$), and the matrix A is in the generalized Liénard form

$$A = \begin{pmatrix} t & -1 & 0 \\ m & 0 & -1 \\ d & 0 & 0 \end{pmatrix},$$

where the coefficients t, m and d are the linear invariants (trace, sum of principal minors and determinant) of the matrix.

Canonical form for 3D relay systems

In short, we have the piecewise linear, discontinuous system

$$(1) \qquad \dot{\mathbf{x}} = F(\mathbf{x}) = \begin{cases} F^-(\mathbf{x}) = A\mathbf{x} + \mathbf{b}, & \text{if} \quad x \le 0, \\ F^+(\mathbf{x}) = A\mathbf{x} - \mathbf{b}, & \text{if} \quad x \ge 0, \end{cases}$$

and the ambiguity in the definition of the vector field will be not relevant as long as we only consider the crossing dynamics.

We have a symmetric discontinuous piecewise linear system with two linearity zones separated by the plane $\Sigma = \{\mathbf{x} \in \mathbb{R}^3 : x = 0\}$

Eigenvalue configuration near a center

We consider eigenvalue configurations for the matrix A near the associated one to a center, that is, we assume that the eigenvalues of A are λ, $\sigma \pm i\omega$ with $\omega > 0$, and analyze the dynamical effects of a sign transition for the real part σ. We suppose so that the linear invariants satisfy

$$
\begin{aligned}
t &= 2\sigma + \lambda, \\
m &= 2\sigma\lambda + \sigma^2 + \omega^2, \\
d &= \lambda\left(\sigma^2 + \omega^2\right).
\end{aligned}
$$

Equilibrium points in 3D relay systems

When $d \neq 0$, or equivalently $\lambda \neq 0$, there appears one equilibrium for each vector field, namely

$$\mathbf{x}_R = (x_R, y_R, z_R) = A^{-1}\mathbf{b} = \left(\frac{b_3}{d}, \frac{t}{d}b_3 - b_1, \frac{m}{d}b_3 - b_2 \right), \quad \mathbf{x}_L = -\mathbf{x}_R.$$

If $b_3 d$ is positive, then both equilibria are real; otherwise, when $b_3 d$ is negative the above equilibria are located out of the half space whose dynamics is ruled by them, and so they are called virtual equilibria.

The invariant focal planes organize the dynamics

There appears an invariant plane passing through each equilibrium point, usually called focal plane. From such focal planes, only the half part with $x < 0$ ($x > 0$) is invariant for F^- (F^+), and so we will speak of invariant focal half-planes. For instance, if $\mathbf{w}$ denotes the left-hand eigenvector associated to the real eigenvalue $\lambda \neq 0$ of the matrix A, then the focal plane for $\mathbf{x}_R$ is

$$\mathbf{w}^\top(\mathbf{x} - \mathbf{x}_R) = 0,$$

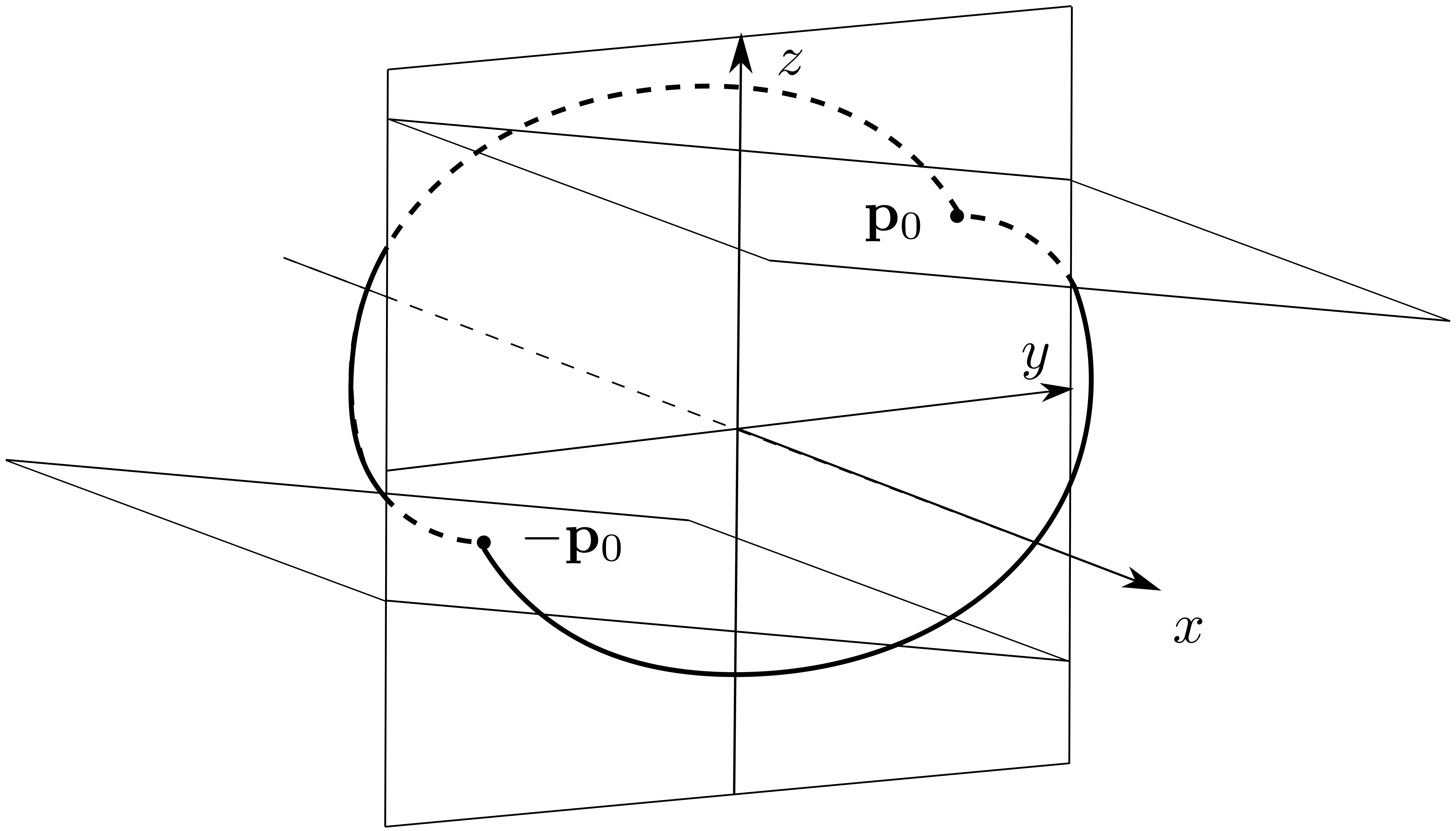
or equivalently,

$$\mathbf{w}^\top\mathbf{x} = \mathbf{w}^\top\mathbf{x}_R = \mathbf{w}^\top A^{-1}\mathbf{b} = \frac{1}{\lambda}\mathbf{w}^\top\mathbf{b}.$$

Effectively, the invariance of such a plane comes from the equality

$$\mathbf{w}^\top\dot{\mathbf{x}} = \mathbf{w}^\top\left(A\mathbf{x} - \mathbf{b}\right) = \lambda\mathbf{w}^\top\mathbf{x} - \mathbf{w}^\top\mathbf{b} = 0,$$

which is true for any point belonging to such a plane.

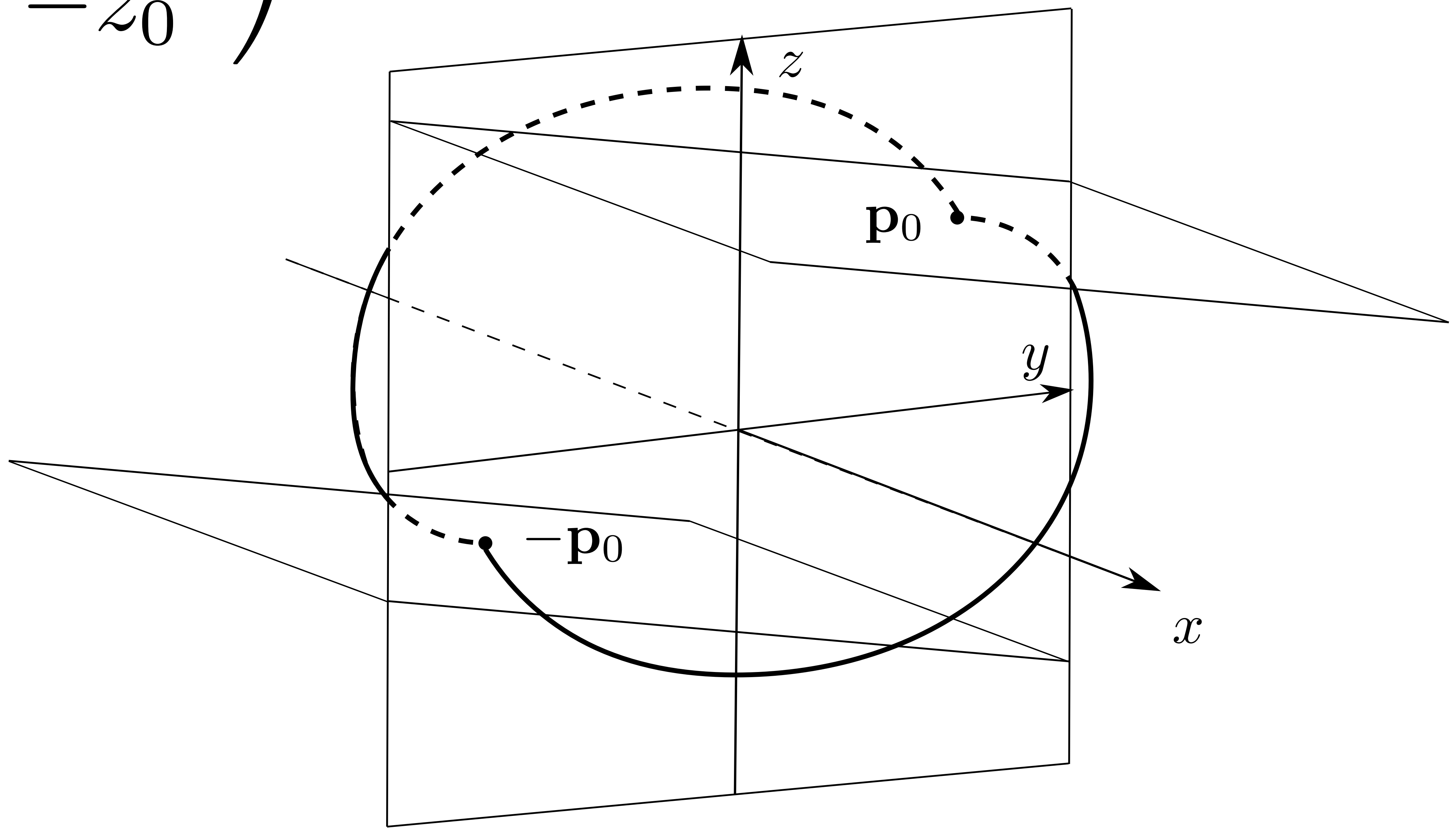


Any symmetric periodic orbit must intersect the discontinuity manifold Σ at two symmetrical points, to be located between the intersections lines of the focal planes with Σ.

The closing equations (using the 'left' semi-orbit)

We can identify symmetric periodic orbits with the solutions of the closing equations

$$e^{A\tau_L}\begin{pmatrix} 0 \\ y_0 \\ z_0 \end{pmatrix} + \int_0^{\tau_L} e^{Au}\mathbf{b}\,du = \begin{pmatrix} 0 \\ -y_0 \\ -z_0 \end{pmatrix}$$



But, how to work near infinity?

Regarding the stability of symmetric periodic orbits…

Proposition If Γ is a symmetric periodic orbit of system (1) with only two transversal crossing points $\mathbf{p}_0 = (0, y_0, z_0) \in \Sigma$ and $\mathbf{p}_1 = -\mathbf{p}_0 \in \Sigma$, and ψ_L, τ_L (ψ_R, τ_R) are the transition map and flight-time function in the left (right) part of the orbit, then $D\psi_R(-y_0, -z_0) = D\psi_L(y_0, z_0)$, $D\tau_R(-y_0, -z_0) = -D\tau_L(y_0, z_0)$, and the two matrices

$$Q_L = \left(\begin{array}{c|c} 1 & D\tau_L(y_0, z_0) \\ \hline \mathbf{0} & D\psi_L(y_0, z_0) \end{array}\right), \quad Q_R = \left(\begin{array}{c|c} 1 & D\tau_R(-y_0, -z_0) \\ \hline \mathbf{0} & D\psi_R(-y_0, -z_0) \end{array}\right)$$

are similar. Moreover, if τ_L is the half-period of the orbit, then we have

$$Q_L = \begin{pmatrix} \frac{1}{b_1+y_0} & 0 & 0 \\ \frac{b_2+z_0}{b_1+y_0} & -1 & 0 \\ \frac{b_3}{b_1+y_0} & 0 & -1 \end{pmatrix} e^{A\tau_L} \begin{pmatrix} b_1 - y_0 & 0 & 0 \\ b_2 - z_0 & -1 & 0 \\ b_3 & 0 & -1 \end{pmatrix}.$$

Some preparatory work: decoupling the Z-dynamics

Proposition The scaling of time, variables and parameters

$$\theta = \omega\tau,\ \tilde{x} = x,\ y = \omega\tilde{y},\ z = \omega^2\tilde{z},\ \lambda = \omega\mu, \sigma = \omega\gamma,\ b_1 = \omega\tilde{b}_1,\ b_2 = \omega^2\tilde{b}_2,\ b_3 = \omega^3\tilde{b}_3,$$

transforms the system into the form

$$\frac{d}{d\theta}\begin{pmatrix}\tilde{x}\\ \tilde{y}\\ \tilde{z}\end{pmatrix} = \begin{pmatrix}2\gamma+\mu & -1 & 0\\ 2\gamma\mu+\gamma^2+1 & 0 & -1\\ \mu\left(\gamma^2+1\right) & 0 & 0\end{pmatrix}\begin{pmatrix}\tilde{x}\\ \tilde{y}\\ \tilde{z}\end{pmatrix} - \operatorname{sgn}(\tilde{x})\begin{pmatrix}\tilde{b}_1\\ \tilde{b}_2\\ \tilde{b}_3\end{pmatrix}.$$

The additional linear change of variables $X = \tilde{x}$, $Y = \tilde{y} - \mu\tilde{x}$, and $Z = \tilde{z} - \mu\tilde{y} + \mu^2\tilde{x}$, produces

$$\frac{d}{d\theta}\begin{pmatrix}X\\ Y\\ Z\end{pmatrix} = \begin{pmatrix}2\gamma & -1 & 0\\ \gamma^2+1 & 0 & -1\\ 0 & 0 & \mu\end{pmatrix}\begin{pmatrix}X\\ Y\\ Z\end{pmatrix} - \operatorname{sgn}(X)\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix},$$

where $\beta_1 = \tilde{b}_1$, $\beta_2 = \tilde{b}_2 - \mu\tilde{b}_1$, and $\beta_3 = \tilde{b}_3 - \mu\tilde{b}_2 + \mu^2\tilde{b}_1$.

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The Z-dynamics is decoupled!

where $\beta_1 = \tilde{b}_1$, $\beta_2 = \tilde{b}_2 - \mu\tilde{b}_1$, and $\beta_3 = \tilde{b}_3 - \mu\tilde{b}_2 + \mu^2\tilde{b}_1$.

Some preparatory work: decoupling the Z-dynamics

For the decoupled system

$$\frac{d}{d\theta}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 2\gamma & -1 & 0 \\ \gamma^2+1 & 0 & -1 \\ 0 & 0 & \mu \end{pmatrix}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} - \operatorname{sgn}(X)\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix},$$

the new focal planes become the horizontal planes $Z = \pm\frac{\beta_3}{\mu}$, and the coordinate Z_0 of the initial point must satisfy

$$-\left|\frac{\beta_3}{\mu}\right| \leq Z_0 \leq \left|\frac{\beta_3}{\mu}\right|,$$

and so any symmetric periodic orbit, including anyone coming from or going to infinity, is bounded in the third coordinate.

Some preparatory work: decoupling the Z-dynamics

Solving separately the boundary value problem

$$\frac{dZ}{d\theta} = \mu Z + \beta_3, \quad Z(0) = Z_0, \quad Z(\theta_L) = -Z_0,$$

we get

$$e^{\mu\theta_L} Z_0 + \frac{1}{\mu}\left(e^{\mu\theta_L} - 1\right)\beta_3 = -Z_0,$$

that is,

$$Z_0 = -\frac{\beta_3}{\mu}\frac{e^{\mu\theta_L} - 1}{e^{\mu\theta_L} + 1} = -\frac{\beta_3}{\mu}\tanh\frac{\mu\theta_L}{2}.$$

The reduced closing equations

For the first two variables, we get

$$(e^{B\theta_L} + I)\left(\begin{pmatrix} -Y_0 \\ 0 \end{pmatrix} + Z_0 BC\right) + (e^{B\theta_L} - I)\left(\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + \beta_3 C\right) = \mathbf{0},$$

where

$$B = \begin{pmatrix} 2\gamma & -1 \\ \gamma^2 + 1 & 0 \end{pmatrix}, \quad C = \frac{1}{1 + (\gamma - \mu)^2}\begin{pmatrix} -1 \\ \mu - 2\gamma \end{pmatrix}.$$

The trick to work near infinity: ultimate closing equations

To work easily near the point at infinity, instead of Y_0, we take $r_0 = 1/Y_0$ as new variable. Consequently, after such substitution and multiplying by r_0, we obtain the reduced closing equations

$$(e^{B\theta_L} + I)\left(\begin{pmatrix} -1 \\ 0 \end{pmatrix} + r_0 Z_0 BC\right) + r_0 \left(e^{B\theta_L} - I\right)\left(\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + \beta_3 C\right) = \mathbf{0}.$$

Lemma The above reduced closing equations with $r_0 = 0$ and $0 < \theta_L < 2\pi$ are only satisfied for $\theta_L = \pi$ and $\gamma = 0$.

An unbounded nonlinear isochronous center!

For $\gamma = 0$ and $\theta_L = \pi$, we have indeed $e^{B\theta_L} + I = \mathbf{0}$, so that, the reduced closing equation becomes

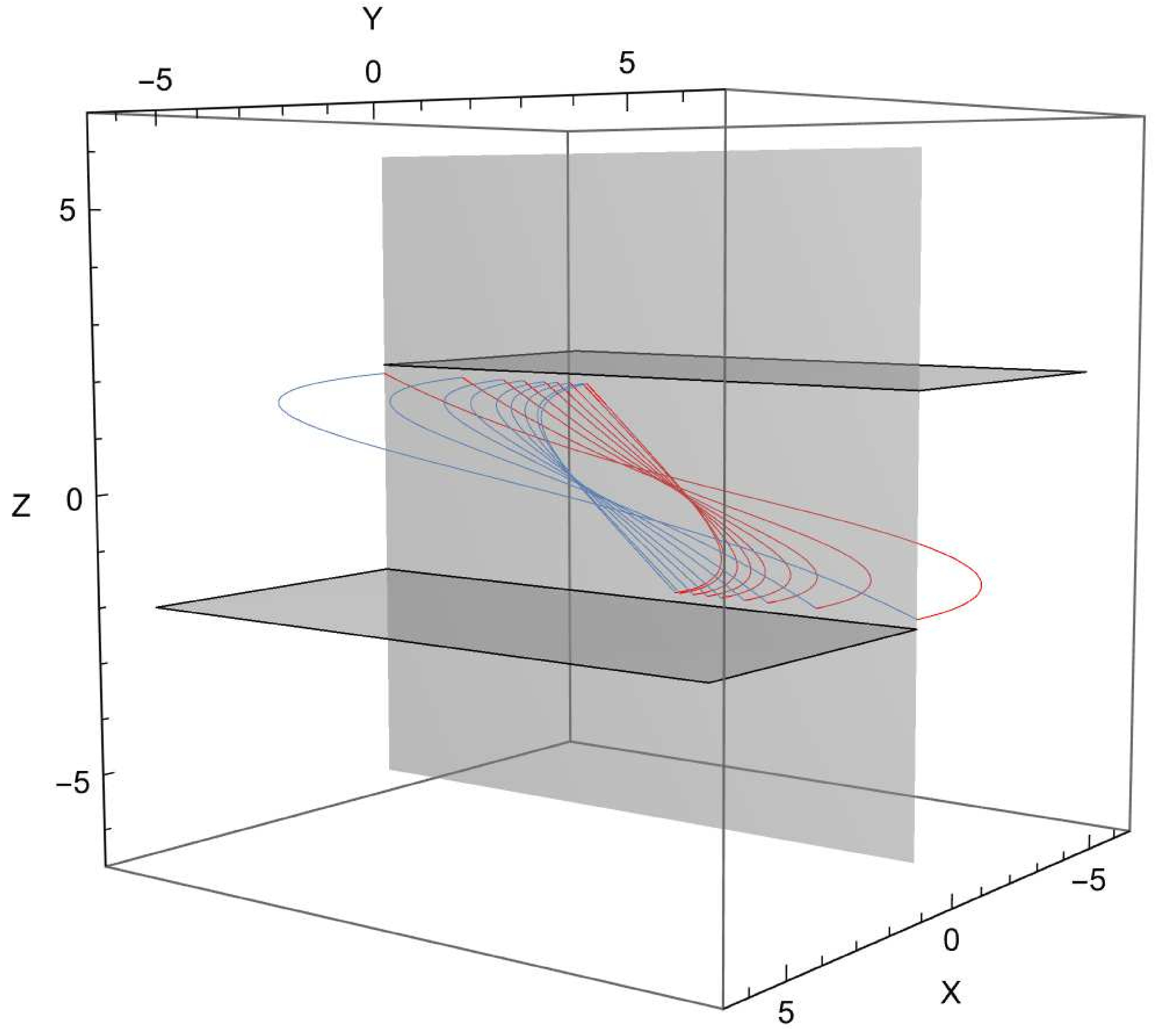
$$-2r_0\left(\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} + \frac{\beta_3}{1+\mu^2}\begin{pmatrix}-1\\ \mu\end{pmatrix}\right) = -\frac{2r_0}{1+\mu^2}\begin{pmatrix}(1+\mu^2)\beta_1 - \beta_3\\ (1+\mu^2)\beta_2 + \mu\beta_3\end{pmatrix} = \mathbf{0}.$$

Therefore, in the most degenerated situation, when the two parameters

$$\tilde{\delta} = \beta_3 - \beta_1(\mu^2+1), \quad \epsilon = \mu\beta_3 + \beta_2(\mu^2+1)$$

vanish, the reduced closed equations are satisfied not only for $r_0 = 0$ but also for any value $r_0 > 0$, all the solutions with $\theta_L = \pi$. This fact indicates that there is an unbounded isochronous center in such a situation.

The unbounded nonlinear isochronous center



Some periodic orbits of the unbounded center when both parameters $\tilde{\delta}$ and ϵ vanish. In the original system, then we have $\delta = b_3 - b_2\lambda - b_1\omega^2 = 0$ and $b_2 = 0$.

The analysis of ultimate closing equations

To apply the implicit function theorem at the point $(r_0, \theta_L, \gamma) = (0, \pi, 0)$, we must study the Jacobian matrix at such point, namely

$$\begin{pmatrix} -2\beta_2 - \frac{2\mu\beta_3}{\mu^2+1} & 1 & 0 \\ 2\beta_1 - \frac{2\beta_3}{\mu^2+1} & 0 & -\pi \end{pmatrix} = \begin{pmatrix} -\frac{2\epsilon}{\mu^2+1} & 1 & 0 \\ -\frac{2\tilde{\delta}}{\mu^2+1} & 0 & -\pi \end{pmatrix},$$

and we conclude that the point is always regular.

After defining $\Upsilon(0) = 1$, $\Upsilon(\xi) = \tanh(\xi)/\xi$ for $\xi \neq 0$, we get

$$\begin{aligned} \theta_L &= \pi + \frac{2\epsilon}{\mu^2+1} r_0 + \frac{8\beta_3\tilde{\delta}}{\pi\left(\mu^2+1\right)^3\mu} {r_0}^2 - \frac{\pi\beta_3\Upsilon\left(\frac{\pi\mu}{2}\right)\left(\tilde{\delta}-\epsilon\mu\right)}{\left(\mu^2+1\right)^2} {r_0}^2 + O\left({r_0}^3\right), \\ \gamma &= \frac{-2\tilde{\delta}}{\pi\left(\mu^2+1\right)} r_0 + \frac{8\tilde{\delta}\left(\epsilon\left(\mu^2+1\right)+\beta_3\mu\right)}{\pi^2\left(\mu^2+1\right)^3} {r_0}^2 - \frac{\beta_3\Upsilon\left(\frac{\pi\mu}{2}\right)\left(\epsilon+\tilde{\delta}\mu\right)}{\left(\mu^2+1\right)^2} {r_0}^2 + O\left({r_0}^3\right). \end{aligned}$$

Main results

Theorem 1 Consider system (1) under the eigenvalue configuration $\{\lambda, \sigma \pm i\omega\}$ with $\omega > 0$, and define the non-degeneracy parameter $\delta = b_3 - b_2\lambda - b_1\omega^2$.
If $\delta \neq 0$, then, for $\sigma = 0$ the system undergoes a Hopf bifurcation from infinity, that is, one symmetric limit cycle of large amplitude appears for $\delta\sigma < 0$ and σ sufficiently small.

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If $\delta \neq 0$, then, for $\sigma = 0$ the system undergoes a Hopf bifurcation from infinity, that is, one symmetric limit cycle of large amplitude appears for $\delta\sigma < 0$ and σ sufficiently small.

The period T of the periodic oscillation is an analytic function at 0, in the variable σ, and the first terms in its series expansion are

$$T = \frac{2\pi}{\omega} - 2\pi\frac{\lambda\delta + \left(\lambda^2 + \omega^2\right) b_2}{\omega^3\delta}\sigma + O\left(\sigma^2\right).$$

Taking $y_0 = 1/r_0 > 0$ as a measure for the amplitude of the bifurcating limit cycle, its series expansion in powers of σ, starts with

$$y_0 = \frac{2\omega\delta}{\pi(\lambda^2 + \omega^2)\sigma} + O\left(1\right).$$

Main results

Theorem 1 Consider system (1) under the eigenvalue configuration $\{\lambda, \sigma \pm i\omega\}$ with $\omega > 0$, and define the non-degeneracy parameter $\delta = b_3 - b_2\lambda - b_1\omega^2$.
If $\delta \neq 0$, then, for $\sigma = 0$ the system undergoes a Hopf bifurcation from infinity, that is, one symmetric limit cycle of large amplitude appears for $\delta\sigma < 0$ and σ sufficiently small.

In particular, when $\lambda \neq 0$, if $\delta > 0$ and $\lambda < 0$, then the bifurcating limit cycle for $\sigma < 0$ is orbitally asymptotically stable. Otherwise, if $\delta < 0$ or $\lambda > 0$ the bifurcating limit cycle is unstable. In the case $\lambda = 0$, assuming $\delta = b_3 - b_1\omega^2 > 0$ a sufficient condition for the stability of the bifurcating limit cycle for $\sigma < 0$ is $b_1 > 0$.

Main results

Theorem 2 Consider system (1) as in Theorem 1, but under the assumptions $b_1 b_2 \neq 0$ and $b_3 = b_2\lambda + \omega^2 b_1$, so that we are in the degenerate case $\delta = 0$.

For $\sigma = 0$ the system still undergoes the Hopf bifurcation at infinity, so that one symmetric limit cycle appears for $b_1 b_2 \sigma < 0$ and σ sufficiently small.

In particular, if $b_1 b_2 > 0$ and $\lambda < 0$ then the limit cycle bifurcates for $\sigma < 0$ and is orbitally asymptotically stable, being unstable if $b_1 b_2 < 0$ or $\lambda > 0$.

For the case $\lambda = 0$, if $b_1 > 0$ and $b_2 > 0$, then the limit cycle bifurcates for $\sigma < 0$ and is orbitally asymptotically stable.

Main results

Theorem 3 Consider system (1) as in Theorem 1, under the assumptions $b_1 b_2 \neq 0$, and b_3 in a neighborhood of the value $b_3^* = b_2\lambda + b_1\omega^2$, so that the non-degeneracy parameter becomes $\delta = b_3 - b_3^*$. Consider a neighborhood of the origin in the parameter plane (σ, δ). The following statements hold.

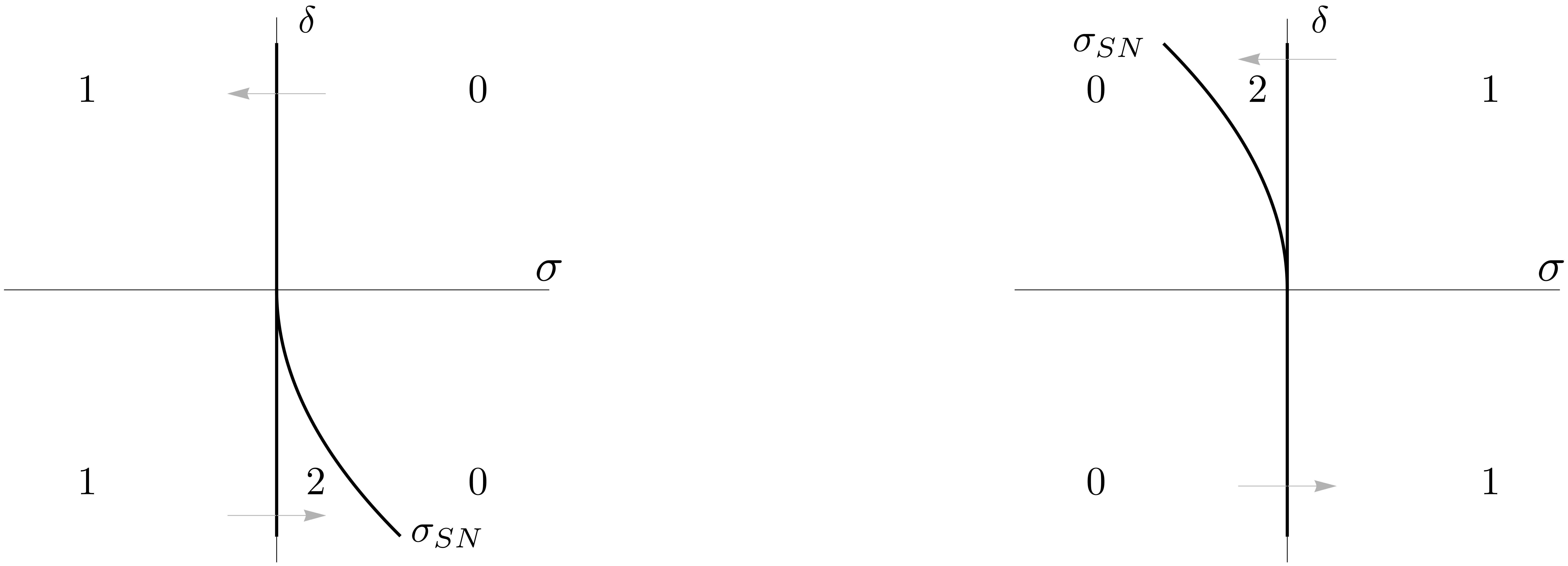
(a) If $b_1 b_2 < 0$ ($b_1 b_2 > 0$), then from the point $(\sigma, \delta) = (0, 0)$ a saddle-node bifurcation curve of periodic orbits emanates in the second quadrant (fourth quadrant) of the parameter plane (σ, δ). This curve is the graph of a function with the following local expression

$$\sigma_{SN}(\delta) = \frac{\omega^2}{\pi^2 \Upsilon\left(\frac{\lambda\pi}{2\omega}\right)(\lambda^2+\omega^2)^2 b_1 b_2}\delta^2 + O(\delta)^3.$$

(b) For (σ, δ) sufficiently close to the origin, if $b_1 b_2 < 0$, $\delta > 0$ and $\sigma_{SN}(\delta) < \sigma < 0$ ($b_1 b_2 > 0$, $\delta < 0$ and $0 < \sigma < \sigma_{SN}(\delta)$), then two limit cycles coexist.

(c) For points at the curve $\sigma = \sigma_{SN}$ there is only one non-hyperbolic limit cycle, and no limit cycles out of the regions indicated in item (b) for $b_1 b_2 < 0$ ($b_1 b_2 > 0$) and with $\sigma < 0$ ($\sigma > 0$).

Main results



Bifurcation set in the parameter plane (σ, δ) for $b_1 b_2 > 0$ (left panel) and $b_1 b_2 < 0$ (right panel) in a neighborhood of the origin. The digits indicate the number of limit cycles associated with the Hopf bifurcation at infinity in each region.

Application to a specific 3D relay family

DiBernardo, Johansson and Vasca (2001), see also DiBernardo et al. (2008), consider a family of relay systems

$$\dot{\mathbf{x}} = \tilde{A}\mathbf{x} + \mathbf{b}u, \quad y = \mathbf{c}^\top \mathbf{x}, \quad u = -\operatorname{sgn} y,$$

where

$$\tilde{A} = \begin{pmatrix} -(2\tilde{\zeta}\tilde{\omega} + \tilde{\lambda}) & 1 & 0 \\ -(2\tilde{\zeta}\tilde{\omega}\tilde{\lambda} + \tilde{\omega}^2) & 0 & 1 \\ -\tilde{\lambda}\tilde{\omega}^2 & 0 & 0 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} \tilde{\kappa} \\ 2\tilde{\kappa}\tilde{\sigma}\tilde{\rho} \\ \tilde{\kappa}\tilde{\rho}^2 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Application to a specific 3D relay family

Assuming in the sequel $\tilde{\kappa} = -1$, the system can be written in the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -(2\tilde{\zeta}\tilde{\omega} + \tilde{\lambda}) & -1 & 0 \\ (2\tilde{\zeta}\tilde{\omega}\tilde{\lambda} + \tilde{\omega}^2) & 0 & -1 \\ -\tilde{\lambda}\tilde{\omega}^2 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \operatorname{sign}(x) \begin{pmatrix} -1 \\ 2\tilde{\sigma}\tilde{\rho} \\ -\tilde{\rho}^2 \end{pmatrix},$$

so that the matrix A is already in our canonical form, and regarding the original parameters in system (1), we have $b_1 = -1$, $b_2 = 2\tilde{\sigma}\tilde{\rho}$, $b_3 = -\tilde{\rho}^2$, $\sigma = -\tilde{\zeta}\tilde{\omega}$, $\lambda = -\tilde{\lambda}$, and $\omega^2 = (1 - \tilde{\zeta}^2)\tilde{\omega}^2$.

Application to a specific 3D relay family

We can select $\tilde{\zeta}$ as the main bifurcation parameter, since at the critical value $\tilde{\zeta} = 0$, we get the configuration $\sigma = 0$, leading to the Hopf bifurcation from infinity. The non-degeneracy parameter for such a bifurcation becomes

$$\delta(\tilde{\zeta}) = -\tilde{\rho}^2 + 2\tilde{\lambda}\tilde{\rho}\tilde{\sigma} + (1 - \tilde{\zeta}^2)\tilde{\omega}^2,$$

and as this parameter depends on the bifurcation parameter $\tilde{\zeta}$, for getting non-degeneracy we will only need

$$\delta_0 = \delta(0) = -\tilde{\rho}^2 + 2\tilde{\lambda}\tilde{\rho}\tilde{\sigma} + \tilde{\omega}^2 \neq 0.$$

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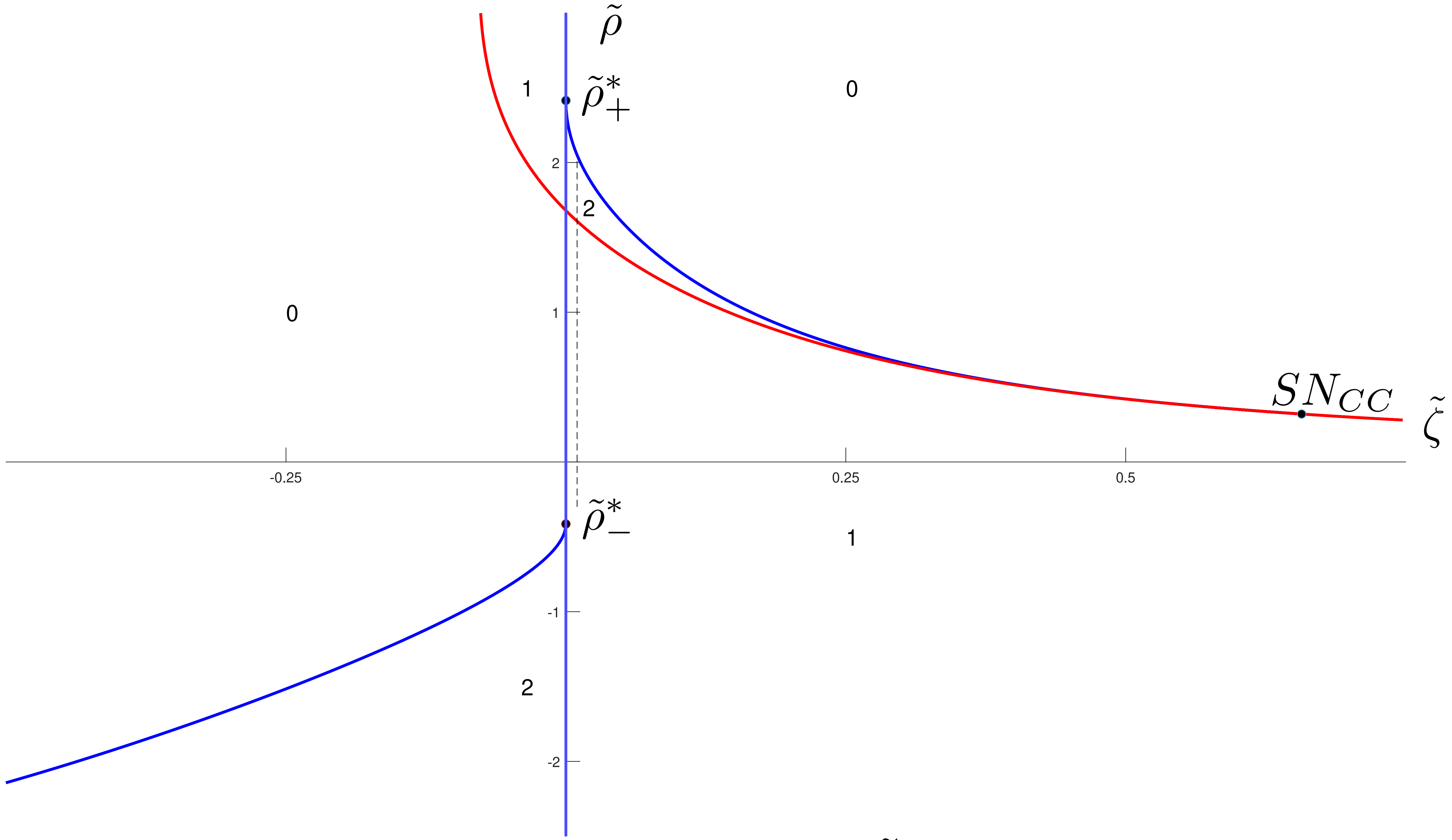
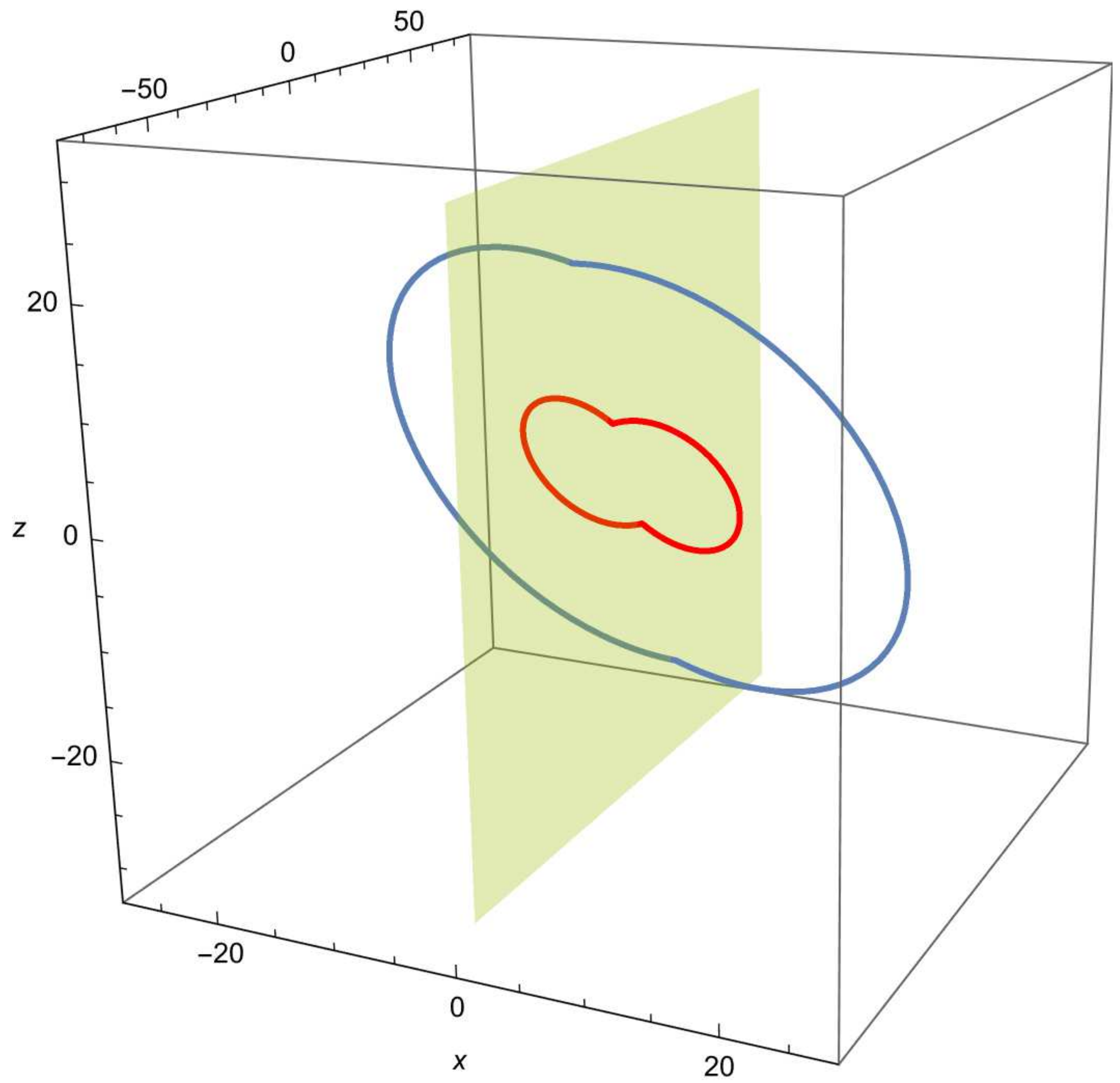
Taking $\tilde{\rho}$ as a second bifurcation parameter, we detect two different critical values where δ_0 vanishes, namely $\tilde{\rho}^*_{\pm} = \tilde{\lambda}\tilde{\sigma} \pm \sqrt{\tilde{\lambda}^2\tilde{\sigma}^2 + \tilde{\omega}^2}$. Thus, we have $\delta_0 = -(\tilde{\rho} - \tilde{\rho}^*_+)(\tilde{\rho} - \tilde{\rho}^*_-)$, so that δ_0 is positive only for $\tilde{\rho}^*_- < \tilde{\rho} < \tilde{\rho}^*_+$. The value $\tilde{\rho}^*_+$ is always positive while $\tilde{\rho}^*_-$ is negative.

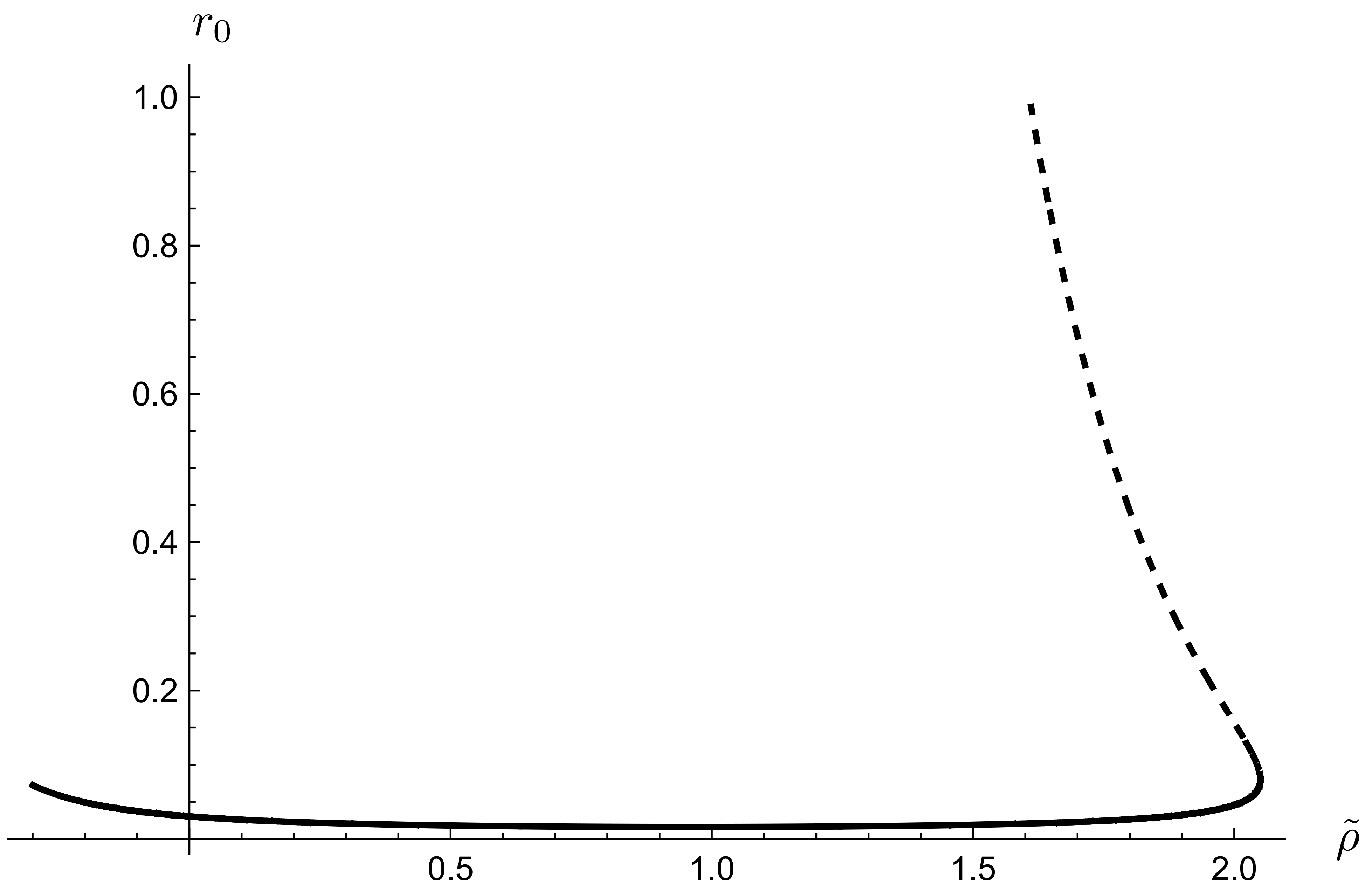
Application to a specific 3D relay family



Bifurcation set of the system in the parameter plane $(\tilde{\zeta}, \tilde{\rho})$ for $\tilde{\kappa} = -1$, and $\tilde{\sigma} = \tilde{\omega} = \tilde{\lambda} = 1$. Apart from the Hopf at infinity bifurcation locus $(\tilde{\zeta} = 0)$, we show the saddle-node bifurcation curves emerging from $(\tilde{\zeta}, \tilde{\rho}^*_+) = (0, 1 + \sqrt{2})$ and $(\tilde{\zeta}, \tilde{\rho}^*_-) = (0, 1 - \sqrt{2})$, and the critical crossing bifurcation curve (in red), which intersects at the point SN_{CC} the saddle-node bifurcation curve that emerges from $(0, \tilde{\rho}^*_+)$, killing it.

Application to a specific 3D relay family





Two limit cycles for $\tilde{\kappa} = -1$, $\tilde{\lambda} = \tilde{\sigma} = \tilde{\omega} = 1$, and $\tilde{\rho} = 2$, near the saddle-node bifurcation that appears at $\tilde{\rho} = 2.05$.

Conclusions

- A complete analysis of the limit cycle bifurcation from infinity in 3D Relay systems, which belong to the class of three-dimensional symmetric discontinuous piecewise linear systems with two zones, is presented.

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- A complete analysis of the limit cycle bifurcation from infinity in 3D Relay systems, which belong to the class of three-dimensional symmetric discontinuous piecewise linear systems with two zones, is presented.
- A criticality parameter is found, whose sign determines the character of the bifurcation. When such non-degeneracy parameter vanishes, a higher co-dimension bifurcation takes place, giving rise to the emergence of a curve of saddle-node bifurcations of periodic orbits, which allows to determine parameter regions where two limit cycles coexist.

Conclusions

- The existence of a large amplitude limit cycle that bifurcates from infinity is justified through a suitable adaptation of the closing equations method, and analytical expressions for its amplitude and period are provided.

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- Derivatives of the corresponding transition maps are rigorously studied in order to characterize the stability of the bifurcating limit cycle.

Conclusions

- The existence of a large amplitude limit cycle that bifurcates from infinity is justified through a suitable adaptation of the closing equations method, and analytical expressions for its amplitude and period are provided.
- Derivatives of the corresponding transition maps are rigorously studied in order to characterize the stability of the bifurcating limit cycle.
- The theoretical results are applied to a specific family of 3D relay systems, where several high co-dimension bifurcation points are detected, organizing the bifurcation set of the family.

I hope you enjoyed the presentation!