Continuous Dynamics of nilpotent polynomial vector fields in $\mathbb{R}^3$

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Motivation

Open Problem: Are there smooth vector fields in $\mathbb{R}^3$ under the hypotheses of the Markus– Yamabe's Problem and having periodic orbits for the system $\dot{x} = F(x)$?

Hypothesis Markus-Yamabe's Problem: Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ a vector field such that:

1. $F(0) = 0$.
2. For all $x \in \mathbb{R}^3$, all the eigenvalues of the Jacobian matrix $JF(x)$ have negative real part.

Vector Fields with The Hypothesis

$$F(x, y, z) = \lambda(x, y, z) + H(x, y, z)$$

where $JH(x, y, z)$ is nilpotent and $\lambda < 0$.

In dimension three, is it possible know which are the maps H such that JH is nilpotent?

- M. Chamberland and A. van den Essen, 2006

$$H = (u(x, y), v(x, y, z), h(u, v)).$$

- D. Yan and M. de Bondt, 2020

$$G = (u_1(x_1, x_2, \ldots, x_n), u_2(x_1, x_2), \ldots, u_n(x_1, x_2)).$$

- Á.C and A. van den Essen, 2020

$$F = (u_1(x_1, x_2), u_2(x_1, x_2, x_3), \ldots, u_{n-1}(x_1, x_2, x_n), u_n(x_1, x_2)).$$

Furthermore, $X + F$ are invertible, thus this large family of maps satisfy the Jacobian Çonjecture.

The polynomial vector field

$$F\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3,\ (x,y,z) \longmapsto (F_1(x,y), F_2(x,y,z), F_3(x,y)), \quad (1)$$

is nilpotent if and only if

$$\begin{aligned} F_1(x,y) &= P_1\big(y + A_1(x)\big), \\ F_2(x,y,z) &= P_2\Big(z + \frac{1}{d_2 p_{d_2}} A_2(x)\Big) - A_1'(x)F_1(x,y), \\ F_3(x,y) &= -\frac{1}{d_2 p_{d2}}\left[-\frac{1}{2}A_1''(x)\big(F_1(x,y)\big)^2 + A_2'(x)F_1(x,y)\right] + A_3, \end{aligned} \quad (2)$$

where

$$\begin{cases} P_i \in \mathbb{R}[s],\ d_i := \deg P_i \geq 1,\ p_{d_i} := \text{the leading coefficient of } P_i, \\ A_1(x) = a_{10} + a_{11}x + a_{12}x^2,\ A_2(x) = a_{20} + a_{21}x,\ A_3 \in \mathbb{R}. \\ \text{If } d_2 > 1,\ \textit{then } A_1''(x) \equiv 0. \end{cases} \tag{3}$$

Consider the differential system

$$\dot{X} = F(X) \tag{4}$$

where $F = (F_1, F_2, F_3)$ as in (2).

Result 1

Each differential system (4) is polynomially integrable. In addition, if $\deg A_1(x) = 1$, *then differential system (4) is polynomially completely integrable.*

Result 2

Assume that $\deg P_1(s) = \deg P_2(s) = 1$ *in system* (4).

1. *If* $\deg A_1(x) = 1$, *then each nontrivial trajectory of system* (4) *goes to infinity in forward and backward time.*
2. *If* $\deg A_1(x) = 2$ *and we define* $\mu := A_3\, a_{12}\, p_{d_2}\, p_{d_1}^2$, *then*
    - 2.1 *each trajectory of* (4) *goes to infinity in forward and backward time if* $\mu > 0$,
    - 2.2 *there exists a unique cuspidal invariant surface* $\mathcal{S}_0$ *of* (4) *and each trajectory of* (4) *in* $\mathbb{R}^3 \backslash \mathcal{S}_0$ *goes to infinity in forward and backward time if* $\mu = 0$,
    - 2.3 *there exists a unique isochronous periodic surface* $\mathcal{S}_\mu$ *of* (4) *and each trajectory of* (4) *in* $\mathbb{R}^3 \backslash \mathcal{S}_\mu$ *goes to infinity in forward and backward time if* $\mu < 0$.

Simpler conjugated systems

By using

$$(x, y, z) \xrightarrow{\Psi} \left(x, y + A_1(x), z + \frac{1}{d_2 p_{d_2}} A_2(x)\right) = (u, v, w), \tag{5}$$

as a change of coordinates, together with equations (2) and (3), the differential system (4) becomes

$$\begin{aligned} \dot{u} &= P_1(v), \\ \dot{v} &= P_2(w), \\ \dot{w} &= \frac{a_{12}}{d_2 p_{d2}} \big(P_1(v)\big)^2 + A_3. \end{aligned} \tag{6}$$

Proof Result 1 (Polynomial Integrability)

The last two equations in (6) form a planar Hamiltonian system, whose Hamiltonian function is

$$G(v,w) := \int P_2(w)\,dw - \frac{a_{12}}{d_2 p_{d2}} \int (P_1(v))^2\,dv - A_3 v.$$

Then, by extending this function to $\mathbb{R}^3$, that is, by defining the polynomial function

$$H(u,v,w) := \int P_2(w)\,dw - \frac{a_{12}}{d_2 p_{d2}} \int (P_1(v))^2\,dv - A_3 v, \qquad (7)$$

we have

$$H_u = 0, \quad H_v = -\frac{a_{12}}{d_2 p_{d2}}(P_1(v))^2 - A_3 \quad \text{and} \quad H_w = P_2(w).$$

Thus,

$$P_1(v)\,H_u + P_2(w)\,H_v + \left(\frac{a_{12}}{d_2 p_{d2}}(P_1(v))^2 + A_3\right) H_w = 0, \quad \forall\,(u,v,w) \in \mathbb{R}$$

We now prove the second part of the theorem. Since $\deg A_1(x) = 1$, $a_{12} = 0$. Then, system (6) reduces to

$$\begin{aligned} \dot{u} &= P_1(v), \\ \dot{v} &= P_2(w), \\ \dot{w} &= A_3. \end{aligned} \tag{8}$$

We have proved that system (6) has a polynomial first integral, then we will show the existence of an additional polynomial first integral of the system.

• If $A_3 = 0$, then (8) admits the two functionally independent polynomial first integrals

$$H_1(u, v, w) = w \quad \text{and} \quad H_2(u, v, w) = \int P_1(v)\, dv - uP_2(w).$$

• If $A_3 \neq 0$, then (8) admits the two functionally independent polynomial first integrals

$$H_1(u,v,w) = \int P_2(w)\,dw - A_3 v$$

and

$$H_2(u,v,w) = A_3^{d_1+1} u - \sum_{j=0}^{d_1} (-1)^j A_3^{d_1-j} \left(\frac{d^j}{dv^j} P_1(v) \right) \xi_j(w),$$

where $\xi_0(w) = w$ and $\xi_j(w) = \int P_2(w) \xi_{j-1}(w)\,dw$ for $j = 1, 2, \ldots, d_1$. ■

Proof Result 2: $\deg P_1(s) = \deg P_2(s) = 1$

Statement 1). Since $\deg A_1(x) = 1$, $a_{12} = 0$. The linear change of coordinates

$$X = \frac{1}{p_{d_1} p_{d_2}}\, u,\ \ Y = \frac{1}{p_{d_1} p_{d_2}}\, P_1(v),\ \ Z = \frac{1}{p_{d_2}}\, P_2(w)$$

transforms the differential system (6), with $a_{12} = 0$, into the differential system

$$\begin{aligned} \dot{X} &= Y, \\ \dot{Y} &= Z, \\ \dot{Z} &= A_3, \end{aligned}$$

which can be solved explicitly. Indeed, the trajectory $\phi_t(X_0, Y_0, Z_0)$ of the system passing through the point (X_0, Y_0, Z_0) has the components:

$$X(t) = \frac{A_3}{6}\, t^3 + \frac{Z_0}{2}\, t^2 + Y_0\, t + X_0,\ Y(t) = \frac{A_3}{2}\, t^2 + Z_0\, t + Y_0,\ Z(t) = A_3\, t + Z_0$$

Statement 2). Since $\deg A_1(x) = 2$, $a_{12} \neq 0$. The linear change of coordinates

$$X = (a_{12}\, p_{d_1})\, u, \;\; Y = (a_{12}\, p_{d_1})\, P_1(v), \;\; Z = (a_{12}\, p_{d_1}^2)\, P_2(w)$$

transforms the differential system (6), with $a_{12} \neq 0$, into the differential system

$$\begin{aligned} \dot{X} &= Y, \\ \dot{Y} &= Z, \\ \dot{Z} &= Y^2 + \mu, \end{aligned} \tag{9}$$

where $\mu = A_3\, a_{12}\, p_{d_2}\, p_{d_1}^2$. Moreover, the first integral (7) for system (6) becomes

$$H(X, Y, Z) = -\mu\, Y + \frac{Z^2}{2} - \frac{Y^3}{3},$$

which is a first integral for system (9). Thus, a trajectory of the system (9) is contained in a level surface $H^{-1}(c) \subset \mathbb{R}^3$ of H, with $c \in \mathbb{R}$.

Since H does not depend on X, $H^{-1}(c)$ has the form

$$H^{-1}(c) = \mathbb{R} \times G^{-1}(c),$$

where $G(Y,Z) = -\mu Y + Z^2/2 - Y^3/3$. Moreover, the last two equations in (9) form the planar Hamiltonian system associated with $G(Y,Z)$.

Case 1: $\mu > 0$. $G(Y,Z)$ does not have any singular point in the YZ-plane. Thus, $G^{-1}(c)$ is homeomorphic to $\mathbb{R}$ for any $c \in \mathbb{R}$. In addition, system (9) does not have singularities in the whole space $\mathbb{R}^3$, then each $H^{-1}(c)$ is a simply connected surface without any singularity of the system. Therefore, each trajectory goes to infinity in forward and backward time.

Case 2: $\mu = 0$. $G(Y, Z)$ has the origin as the unique singularity in the YZ-plane. In fact, $(0, 0)$ is a cusp singularity of $G(Y, Z)$. Since $G(0, 0) = 0$, $G^{-1}(0)$ is the cuspidal cubic curve. Hence, $G^{-1}(c)$ is homeomorphic to $\mathbb{R}$ for any $c \neq 0$. In addition, since all the singularities of (9) are of the form $(X, 0, 0)$, they are contained in the cuspidal invariant (singular) surface $S_0 := H^{-1}(0) = \mathbb{R} \times G^{-1}(0)$. This implies that $H^{-1}(c)$, with $c \neq 0$ is a simply connected surface without any singularity of the system. Hence, all trajectories in $\mathbb{R}^3 \backslash S_0$ have to escape to infinity in forward and backward time.

Case 3: $\mu < 0$. We can change the parameter μ by $-\beta^2$, with $\beta > 0$. Then, by using the linear the change of coordinates $X = \sqrt{\beta}x$, $Y = \beta(y-1)$, $Z = \beta^{3/2} z$ and the linear change of time $\tau = \sqrt{\beta}\, t$, the differential system (9), with $\mu = -\beta^2$, is transformed into the differential system

$$\begin{aligned} x' &= y - 1, \\ y' &= z, \\ z' &= y(y-2), \end{aligned} \tag{10}$$

where the prime denotes the derivative with respect to a new time variable τ.

(10) has a unique isochronous periodic surface

The differential system (10) does not have any singularity in the whole $\mathbb{R}^3$ and it has the polynomial first integral

$$H(x, y, z) = (6y^2 + 3z^2 - 2y^3)/6.$$

Since this first integral does not depend on x,
$H^{-1}(c) = \mathbb{R} \times G^{-1}(c)$, where $G(y, z) = (6y^2 + 3z^2 - 2y^3)/6$.
The last two equations in (10) form, in the yz-plane, the planar Hamiltonian system associated with $G(y, z)$, whose singularities are $(0, 0)$ and $(2, 0)$. A simple computation shows that they are a center and a saddle, respectively.

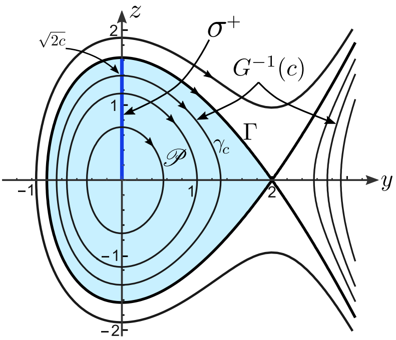
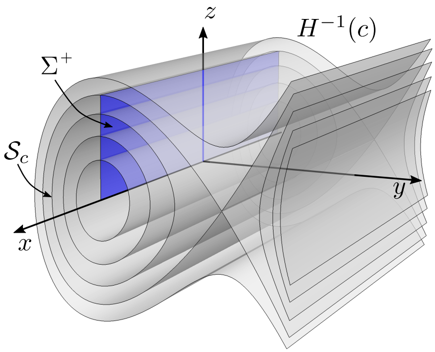


Figure: *a*) Phase portrait of the planar Hamiltonian system associated with $G(y, z)$.
b) σ^+ transversal section



b)

Figure: *b*) Foliation of the first integral of (10).

•This Hamiltonian system has a period annulus $\mathscr{P}$ surrounding the center $(0,0)$ and bounded by the homoclinic loop Γ that joins the stable and the unstable manifolds of the saddle point $(2,0)$.

• Since $G(0,0) = 0$ and $G(2,0) = 4/3$, for all $c \in (0,4/3)$ the level curve $G^{-1}(c)$ has a connected component γ_c homeomorphic to the unit circle $\mathbb{S}^1$ that forms part of $\mathscr{P}$ and the level surface $H^{-1}(c)$ has a connected component $\mathcal{S}_c$ homeomorphic to the cylinder $\mathbb{R} \times \mathbb{S}^1$.

• The straight lines $L_0 := \mathbb{R} \times \{(0,0)\}$ and $L_2 := \mathbb{R} \times \{(2,0)\}$ are invariant by the flow of (10). Thus, as trajectories, they go to infinity in forward and backward time.

Moreover, a straightforward analysis on the topology of $G^{-1}(c)$ implies that for any $c \in \mathbb{R}$,

$$H^{-1}(c) \cap \left(\mathbb{R}^3 \setminus \left(\cup_{c \in (0,4/3)} \mathcal{S}_c \cup L_0 \cup L_2 \right)\right)$$

is formed only by disjoint simply connected surfaces. Hence:

i) only the invariant surfaces $\mathcal{S}_c$, with $c \in (0, 4/3)$, could support periodic orbits and

ii) any trajectory of system (10) in $\mathbb{R}^3 \setminus \cup_{c \in (0,4/3)} \mathcal{S}_c$ goes to infinity in forward and backward time.

It remains to prove the existence of only one surface $S^* = \mathcal{S}_{c^*}$, with $c^* \in (0, 4/3)$, that is foliated by periodic orbits of the same period.

- There exists a well-defined Poincaré first return map

$$\begin{array}{rcl} \mathcal{P}\colon \Sigma^+ & \longrightarrow & \Sigma^+ \\ (x,c) & \longmapsto & \phi_{\tau(x,c)}(x,c), \end{array}$$

where $\tau(x,c)$ is the time of first return of the point (x,c) to Σ^+.

- There exist a unique c^* such that $\mathcal{P}(x,c^*) = (x,c^*)$.

Moreover, $\mathcal{P}(x, c) = \phi_{\tau(x,c)}(x, c) = (x_c(\tau(x, c)), c)$, which implies that the fixed points of $\mathcal{P}$ are in correspondence with the zeros of the displacement function

$$L(x, c) := x_c(\tau(x, c)) - x_c(0).$$

Since the right-hand side of the system

$$\begin{aligned} x' &= y - 1, \\ y' &= z, \\ z' &= y(y - 2), \end{aligned} \tag{11}$$

does not depend on x, the time of first return $\tau(x, c)$ does not either, that is, $\tau(x, c) = \tau(0, c)$.

• Thus, if $L(0, c^*) = 0$, then $L(x, c^*) = 0$ for all $x \in \mathbb{R}$, whence $\mathcal{S}_{c^*}$ will be a isochronous (periodic) surface,

Uniqueness of the isochronous surface $\mathcal{S}_{c^*}$.

• It is enough to study the function

$$L(0,c) = x_c(\tau(0,c)) - x_c(0), \quad \text{with } x_c(0) = 0.$$

• It proves that $L(0,c) < 0$ for $0 < c \leq 2/3$, $L(0,c) > 0$ for $2/3 \ll c < 4/3$, and $L(0,c)$ is a monotonous increasing function in $\big(2/3, 4/3\big)$, which implies the existence of a unique $c^* \in (0, 4/3)$ such that $L(0, c^*) = 0$.

Question

- Is any periodic orbit in $\mathcal{S}_c$ persisting under the perturbation λI with $\lambda < 0$?

A positive answer to the this question would give a affirmative response to the initial open problem.

We note that for $d_1 > 1$ the planar Hamiltonian system associated with system (4) can have several period annuli. For instance, by taking $P_1(s) = s^2 - s - 3$, $P_2(s) = s$, $a_{12} = 1$ and $A_3 = -6$, the system (4) has two period annuli. Hence, we can ask:

- How many periodic surfaces can have system (4) for $d_1 > 1$ and $d_2 = 1$?

Thank you very much!!!!