Center-focus problem by its complex separatrices

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Introduction

We consider families of real analytic planar differential systems

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or equivalently planar vector fields

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- $(x, y) = (0, 0)$ is a monodromic singularity of $\mathcal{X}$, that is local orbits turn around the origin for any $\lambda \in \Lambda \subset \mathbb{R}^p$.
- Since $\mathcal{X}$ is analytic, independently I'lyashenko and Écalle, prove that the singularity only can be either a *center* or a *focus*.

Poincaré-Lyapunov center-focus problem

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To discern the subsets of Λ corresponding to a center and a focus.

Degrees of degeneracy in the center-focus problem

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- DEGENERATE CASE: When $D\mathcal{X}(0,0) \equiv 0$ the center-focus problem remains open except few specific cases.

Real analytic invariant curves from complex separatrices

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REMARK: We are only interested in invariant curves $F(x, y) = 0$ passing through the origin, that is with $F(0, 0) = 0$.
This is because $U(x, y)F(x, y) = 0$ is also an invariant analytic curve of $\mathcal{X}$ for any analytic unit $U(x, y)$ with $U(0, 0) \neq 0$

Real analytic invariant curves from complex separatrices

Toy example

- Linear vector field $\mathcal{X} = (-y + \lambda x)\partial_x + (x + \lambda y)\partial_y$ with $\lambda \in \mathbb{R}$.

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- Real analytic invariant curve $F^{\mathbb{R}}(x, y) = f_1(x, y)f_2(x, y) = x^2 + y^2 = 0$ with cofactor $K^{\mathbb{R}}(x, y) = K_1(x, y) + K_2(x, y) = 2\lambda$.

Existence of real analytic invariant curves at monodromic singularities

Theorem 1

- Let $\mathcal{X} = P(x,y)\partial_x + Q(x,y)\partial_y$ be real analytic planar vector field in a neighborhood of a monodromic singularity at the origin;

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Sketch of the proof: We take the "canonical complexification" $\mathcal{X}^{\mathbb{C}}$ at $(\mathbb{C}^2, 0)$ of the real analytic vector field $\mathcal{X}$ at $(\mathbb{R}^2, 0)$ and next we use Camacho-Sad separatrix theorem.

The Newton diagram of $\mathcal{X}$

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$$P(x,y) = \sum_{(i,j)\in\mathbb{N}^2} a_{ij} x^i y^{j-1}, \quad Q(x,y) = \sum_{(i,j)\in\mathbb{N}^2} b_{ij} x^{i-1} y^j,$$

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- The Newton diagram $\mathbf{N}(\mathcal{X})$ of $\mathcal{X}$ is the boundary of the convex hull of the set

$$\bigcup_{(i,j)\in\mathrm{supp}(\mathcal{X})} \{(i,j) + \mathbb{R}^2_+\}.$$

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$W(\mathbf{N}(\mathcal{X})) \subset \mathbb{N}^2$ is the set containing all the weights associated to the edges in $\mathbf{N}(\mathcal{X})$.

The weighted polar blow-up

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Given $(p, q) \in W(\mathbf{N}(\mathcal{X}))$, we take the blow-up $(x, y) \mapsto (\rho, \varphi)$ given by

$$x = \rho^p \cos\varphi, \quad y = \rho^q \sin\varphi. \tag{2}$$

The differential equation on the cyclinder C

In coordinates (ρ, φ) $\mathcal{X}$ is orbitally equivalent to

$$\dot{\rho} = R(\varphi, \rho) = \rho F_r(\varphi) + O(\rho^2), \quad \dot{\varphi} = \Theta(\varphi, \rho) = G_r(\varphi) + O(\rho).$$

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We define the (p, q)-characteristic directions at the origin of $\mathcal{X}$ as:

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We consider the ordinary differential equation:

$$\frac{d\rho}{d\varphi} = \mathcal{F}(\varphi, \rho) = \frac{R(\varphi, \rho)}{\Theta(\varphi, \rho)}, \tag{3}$$

where $\mathcal{F} : C \backslash \Theta^{-1}(0) \to \mathbb{R}$ being the cylinder

$$C = \{(\varphi, \rho) \in \mathbb{S}^1 \times \mathbb{R} : 0 \leq \rho \ll 1\} \text{ with } \mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$$

The invariant curve on the cyclinder C

Let $F(x, y) = 0$ be a real invariant analytic curve of $\mathcal{X}$ with $F(0, 0) = 0$ (which always exists by Theorem 1). Then

$$\mathcal{X}(F) = KF.$$

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- $\hat{\mathcal{X}}=\partial_\varphi+\mathcal{F}(\varphi,\rho)\partial_\rho$
- $\hat{F}(\varphi,\rho)=F(\rho^p\cos\varphi,\rho^q\sin\varphi)$;
- $\hat{K}$ is the cofactor of the invariant curve $\hat{F}=0$ of $\hat{\mathcal{X}}$.

The cofactor of the invariant curve on the cyclinder C

The explicit expression of $\hat{K}$ is:

$$\hat{K}(\varphi, \rho) = \frac{D(\varphi) K(\rho^p \cos\varphi, \rho^q \sin\varphi)}{\rho^r \Theta(\varphi, \rho)}.$$

- $D(\varphi) = p\cos^2\varphi + q\sin^2\varphi > 0$

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- $D(\varphi) = p\cos^2\varphi + q\sin^2\varphi > 0$
- r is the leading (p, q)-quasihomogeneous degree in the expansion

$$\mathcal{X} = \sum_{j \geq r} \mathcal{X}_j$$

with $\mathcal{X}_j$ the (p, q)-quasihomogeneous vector field of degree j.

The *Cauchy principal value* of an improper integral

Given a continuous function f defined in $I \subset [0, 2\pi] \backslash \Omega$ with $\Omega = \{\theta_1^*, \ldots, \theta_\ell^*\}$, the Cauchy principal value of the integral $\int_I f(\theta)\, d\theta$ is defined as

$$PV \int_I f(\theta)\, d\theta = \lim_{\varepsilon \to 0^+} \int_{I_\varepsilon} f(\theta)\, d\theta,$$

when the limit exists. Here we have used the notation $I_\varepsilon = I \backslash J_\varepsilon$ with $J_\varepsilon = \cup_{i=1}^{\ell} (\theta_i^* - \varepsilon, \theta_i^* + \varepsilon)$.

The main result

Let $\rho(\varphi; \rho_0)$ be the solution of the Cauchy problem

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Theorem 2

Let $F = 0$ be an analytic invariant curve of $\mathcal{X}$ through the origin. For any initial condition $\rho_0 > 0$ sufficiently small, $I_{\hat{K}}(\rho_0)$ exists and moreover the origin is a center if and only if $I_{\hat{K}}(\rho_0) \equiv 0$.

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REMARK: If F is a first integral $\Longrightarrow \hat{K} \equiv 0 \Longrightarrow I_{\hat{K}}(\rho_0) \equiv 0$.

Overcoming the difficulty of computing $\rho(\varphi; \rho_0)$

Corollary (sufficient focus condition)

Assume the cofactor K of an analytic invariant curve through the origin has the (p, q)-quasihomogeneous expansion

$$K(x, y) = K_{\bar{r}}(x, y) + \cdots .$$

If $K_{\bar{r}}(\cos\varphi, \sin\varphi)$ is a semi-definite function in $\mathbb{S}^1$ then the origin is a focus of $\mathcal{X}$.

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How to compute $K_{\bar{r}}(x,y)$ without the expression of F ?

Computing $K_{\bar{r}}(x, y)$

In order to compute $K_{\bar{r}}(x, y)$ we could apply several methods:

Newton-Puiseux factorization

By Newton-Puiseux Theorem there exists a finite factorization

$$F^{\mathbb{R}}(x, y) = u(x, y) \prod_i (y - y_i^*(x)) \tag{4}$$

- u is a real analytic unit $u(0, 0) \neq 0$;
- $y_i^*(x)$ are complex holomorphic functions of x^{1/n_i} with $y_i^*(0) = 0$ called *branches* of $F^{\mathbb{R}}$ at the origin;
- The exponents $n_i \in \mathbb{Z}^+$ are called the *indices* of the branches y_i^*.

Computing $K_{\bar{r}}(x, y)$

Invariant branching theory (Bruno)

- The invariant branches are $y_i^*(x) = \alpha_0 x^{q/p} + \cdots$ with $(p, q) \in W(\mathbf{N}(\mathcal{X}))$;
- α_0 is computed using that $y^p - \alpha_0 x^q = 0$ is an invariant algebraic curve of $\mathcal{X}_r$.
- The branches have the expansion

$$y_i^*(x) = \sum_{j \geq 0} \alpha_j x^{\frac{q}{p} + \frac{j}{n_i}},$$

- There are general methods to compute the index n_i (Fuchs indices, etc...).

Computing $K_{\bar{r}}(x, y)$

We consider the (p, q)-quasihomogeneous expansions:

$$\begin{aligned} \mathcal{X} &= \mathcal{X}_r + \cdots , \\ F(x, y) &= F_s(x, y) + \cdots , \\ K(x, y) &= K_{\bar{r}}(x, y) + \cdots . \end{aligned}$$

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Direct method (Algaba et. al.)

- $F_s = 0$ is an invariant algebraic curve of $\mathcal{X}_r$ with cofactor $K_{\bar{r}}$.
- The irreducible factors of F_s are factors of the inverse integrating factor $V(x,y) = (px, qy) \wedge \mathcal{X}_r$ of $\mathcal{X}_r$.

Example: Mañosas monodromic family

Victor Mañosas shows that family

$$\dot{x} = xy^2 - y^3 + ax^5, \quad \dot{y} = 2x^7 - x^4 y + 4xy^2 + y^3, \qquad (5)$$

has a monodromic singularity at the origin with parameters $\Lambda = \{a \in \mathbb{R} : \Delta(a) := 32 - (1 + 3a)^2 > 0\}$. Moreover he proves:

Mañosas family in Λ

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MAÑOSAS PROOF:

i) The Poincaré map is $\Pi(x) = \eta_1 x + o(x)$ with

$$\eta_1 = \exp\left(\pi + \frac{4\pi a}{\sqrt{\Delta(a)}}\right) \neq 1 \text{ if } a \neq -31/25. \qquad (6)$$

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ii) When $a = -31/25$ he uses a Lyapunov function.

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- Using Bruno's theory we check if there is an invariant branch of the form $y_j^*(x) = \alpha_0 x^{1/1} + o(x)$ with $\alpha_0 \neq 0$.
  - The leading term α_0 is computed imposing that $y^1 - \alpha_0 x^1 = 0$ is an invariant curve of $\mathcal{X}_2$

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- Taking the weights $(p, q) = (1, 1)$ we see that $\mathcal{X} = \mathcal{X}_2 + \cdots$ with $\mathcal{X}_2 = (xy^2 - y^3)\partial_x + (4xy^2 + y^3)\partial_y$;
- Using Bruno's theory we check if there is an invariant branch of the form $y_j^*(x) = \alpha_0 x^{1/1} + o(x)$ with $\alpha_0 \neq 0$.
  - The leading term α_0 is computed imposing that $y^1 - \alpha_0 x^1 = 0$ is an invariant curve of $\mathcal{X}_2 \Longrightarrow \alpha_0 = \pm i\sqrt{2} \in \mathbb{C}$;

Example: Our proof in Mañosas monodromic family

Mañosas family in Λ

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  - Now we know that the invariant branches of $\mathcal{X}$ at the origin are

$$y_j^*(x) = \alpha_0 x^{\frac{1}{1}} + \sum_{i \geq 1} \alpha_i x^{\frac{1}{1} + \frac{i}{n_j}}$$

for some index $n_j \in \mathbb{Z}^+$.

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- $F(x, y) = (y - y_1^*(x))(y - y_2^*(x)) = 0$ is a real analytic invariant curve of $\mathcal{X}$ through the origin;

Example: Our proof in Mañosas monodromic family

- We get the $(1,1)$-quasihomogeneous expansions:

$$\begin{aligned} F(x,y) &= F_2(x,y) + \cdots = 4x^2 + y^2 + \cdots , \\ K(x,y) &= K_2(x,y) + \cdots = 2y^2 + \cdots . \end{aligned}$$

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- Clearly $K_2(\cos\varphi, \sin\varphi)$ is semi-positive defined.

Example 2

We consider the family of vector fields

$$\begin{aligned} \dot{x} &= \lambda_1(x^6+3y^2)(-y+\mu x)+\lambda_2(x^2+y^2)(y+Ax^3), \\ \dot{y} &= \lambda_1(x^6+3y^2)(x+\mu y)+\lambda_2(x^2+y^2)(-x^5+3Ax^2y). \quad (7) \end{aligned}$$

The $(0,0)$ is monodromic if and only if the parameters lie in

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(i) If $\mu \neq 0$ then the origin is a focus;

(ii) If $\mu = 0$ then the origin is a focus or a center according to whether $A \neq 0$ or $A = 0$, respectively.

Proof

(i) The full family has two invariant curves

$$F_1(x,y) = x^2 + y^2 = 0, \quad F_2(x,y) = y^2 + x^6/3 = 0,$$

with associated cofactors

$$\begin{aligned} K^{(1)}(x,y) &= 2(\lambda_2 xy(1-x^4) + A\lambda_2 x^2(x^2+3y^2) + \lambda_1\mu(x^6+3y^2), \\ K^{(2)}(x,y) &= 6(\lambda_1 xy(1-x^4) + A\lambda_2(x^4+x^2y^2) + \lambda_1\mu(x^6+y^2)). \end{aligned}$$

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(ii) $W(\mathbf{N}(\mathcal{X})) = \{(1,1),(1,3)\}$ and leading parts are

- $(p,q) = (1,1)$ and $\mathcal{X}_2 = *\partial_x + \lambda_1 3y^2(x+y\mu)\partial_y$;
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Consequently, $\Omega_{11} \neq \emptyset$ and $\Omega_{13} \neq \emptyset$.

Proof

- We take the invariant curve $F = F_1^{m_1} F_2^{m_2} = 0$ with arbitrary $m_i \in \mathbb{Z}^+$ whose cofactor is $K = m_1 K^{(1)} + m_2 K^{(2)}$

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(i) Under the conditions in statement (i) the function $K_2(\cos\varphi, \sin\varphi)$ is sign-defined in $\mathbb{S}^1$.

(ii) Under the conditions in statement (ii), $K_2(x,y) \equiv 0$ and $K(x,y) = K_4(x,y)$ such that $K_4(\cos\varphi, \sin\varphi)$ is sign-defined in $\mathbb{S}^1$ when $A \neq 0$ and $K(x,y) \equiv 0$ when $A = 0$.

Proof

- We take the invariant curve $F = F_1^{m_1} F_2^{m_2} = 0$ with arbitrary $m_i \in \mathbb{Z}^+$ whose cofactor is $K = m_1 K^{(1)} + m_2 K^{(2)}$
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REMARK: Taking the $(1,3)$-quasihomogeneous expansion of K we get no new results.

MANY THANKS

FOR YOUR ATTENTION !!