# Non-hyperbolic slow-fast systems and chaotic dynamics in a 3d predator-prey system

P. De Maesschalck
Joint work with Y. Patsios

February 2023

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contains traces of finished and/or ongoing works
with M. Alvarez, J. Torregrosa, X. Zhang

## The basics

**An elementary slow-fast system**

$$\begin{cases} \dot{x} &= \epsilon \\ \dot{y} &= a(x)y \end{cases}$$

Orbit through $(x_0, y_0)$ is expressed as a graph

$$y = y_0 \exp \frac{1}{\epsilon} \int_{x_0}^{x} a(s)ds$$

The integral is called a slow divergence integral. Suppose now that

$$\begin{cases} a(x) < 0 & x < x_* \\ a(x) > 0 & x > x_* \end{cases}$$

Then we have an implicitly defined transition map

$$x_0 \mapsto x_1 \text{ with } \int_{x_0}^{x_1} a(s)ds = 0$$

that takes a point from $\{y = y_0, x < x_*\}$ to a point in $\{y = y_0, x > x_*\}$.

Suppose $a(x) = \arctan x$. Then by symmetry we should have

$$\int_{x_0}^{-x_0} a(s)ds = 0$$

So the entry-exit map is trivial

$$x_0 \mapsto -x_0$$

The numerics is less trivial. Let us consider the o.d.e. integrator from

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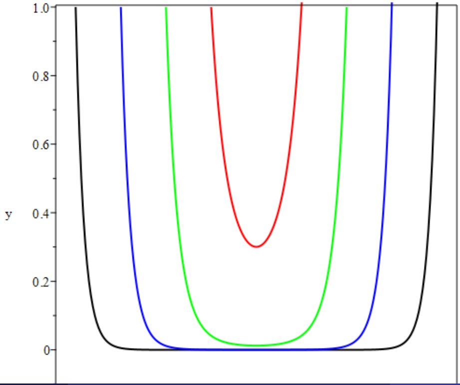

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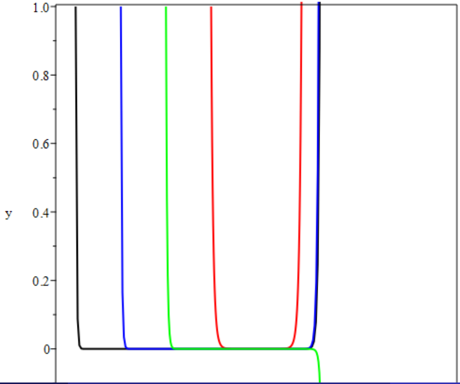

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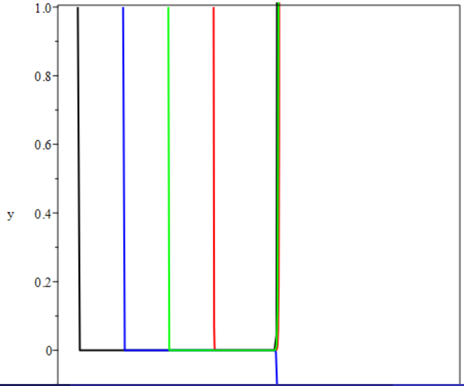
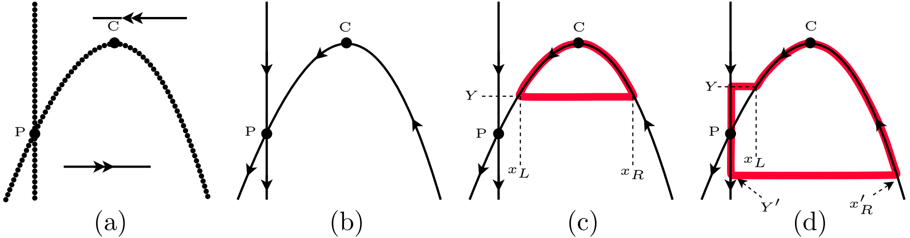


# The basics in a Rosenzweig-Macarthur predator-prey model

$$\begin{cases} \dot{x} = x(1-x) - \dfrac{xy}{\lambda + x}, \\ \dot{y} = \epsilon y\left(-\mu + \dfrac{x}{\lambda + x}\right), \end{cases}$$



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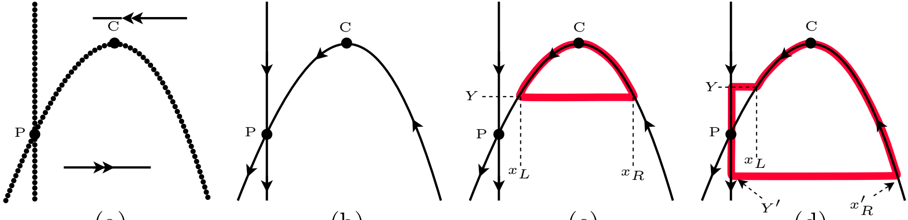


(a) (b) (c) (d)

*Though predator-prey cycles like in (d) are possible theoretically, the number of prey drops to exponentially small levels, even for moderate values of* $\epsilon$ *!*

## From hyperbolic to non-hyperbolic

Consider the slow-fast system

$$\begin{cases} \dot{x} &= \epsilon \\ \dot{y} &= a(x)y^2 \end{cases}$$

Orbit through $(x_0, y_0)$ is expressed as a graph

$$y = y_0 \left(1 - \frac{y_0}{\epsilon} \int_{x_0}^{x} a(s)ds\right)^{-1}$$

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The rest is the same!

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In this case even Maple can integrate numerically!

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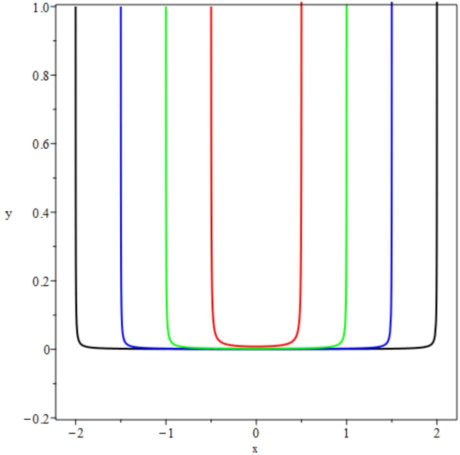

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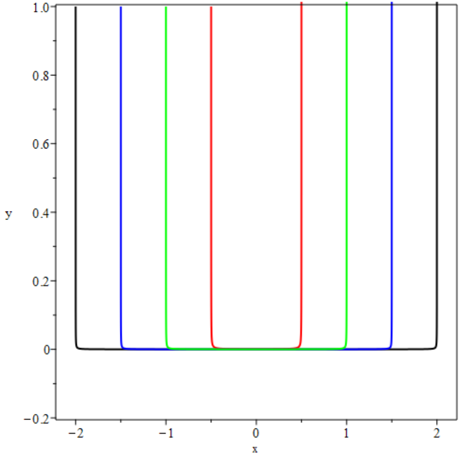
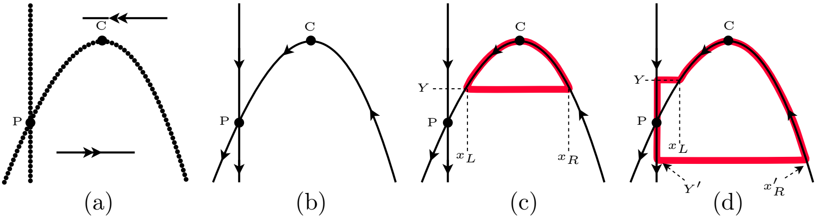


## Non-hyperbolic Rosenzweig-Macarthur predator-prey model

$$\begin{cases} \dot{x} &= x^2(1-x) - \dfrac{x^2 y}{\lambda + x}, \\ \dot{y} &= \epsilon y \left( -\mu + \dfrac{x}{\lambda + x} \right), \end{cases}$$

- Slow-fast analysis in first quadrant stays the same




- This adaptation looks a bit like changing the Holling type of functional response but it still is a bit different than that.
- Population numbers vary in a more realistic way in this adapted model.

## Going beyond the elementary examples

$$\begin{cases} \dot{x} &= \epsilon f(x, y, \epsilon, \lambda) \\ \dot{y} &= a(x, y, \epsilon, \lambda) y \end{cases}$$

$\Rightarrow$ studied by many people (Pontryagin, Benoit, Liu, ...)

In [DM & Schecter 2016] We reconsidered it in a unified way together with

$$\begin{cases} \dot{x} &= \epsilon f(x, y, \epsilon, \lambda) \\ \dot{y} &= a(x, y, \epsilon, \lambda) y^2 \end{cases}$$

## Theorem

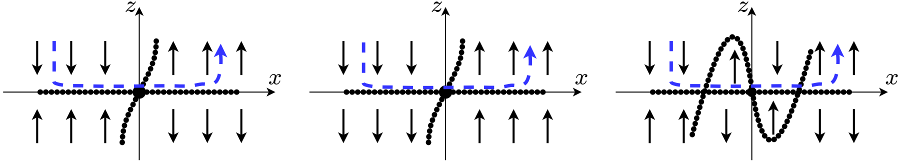
*Let $x_0$ and $x_1$ be such that*

$$\int_{x_0}^{x_1} a(x, 0, 0, \lambda)/f(x, 0, 0, \lambda) dx = 0$$

*Then for $y_0 > 0$ small enough there is a well-defined entry-exit map $\{y = y_0\} \to \{y = y_0\}$ near $x_0$ given by*

$$x \mapsto P(x, \epsilon, \epsilon \log \epsilon), \qquad \text{with } P(x_0, 0, 0) = x_1.$$

The entry–exit in [DM-Schecter 2016] even allows multiple turning points:




However:

- we did not yet treat the case of dimension $> 2$. Ongoing research in a joint project with X. Zhang
- we did not treat the boundaries of the entry and exit sections: what if the asymptotic entry or exit point is exactly at a turning point? Ongoing research in a joint project with M. Alvarez in view of the study of canard-type solutions to Abel equations
- we did not discuss saddle-node type unfolding of the double critical curve Preprint with J. Torregrosa dealing with limit cycles and critical periods (at present in the plane only)

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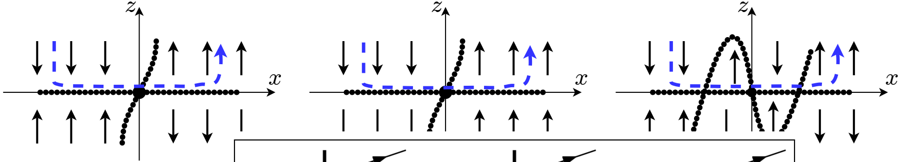


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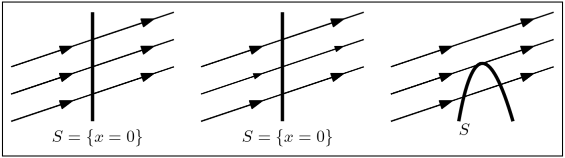


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## Back to the predator-prey model

$$\begin{cases} \dot{x} &= x(1-x) - \dfrac{xy}{\lambda + x}, \\ \dot{y} &= \epsilon y \left( -\mu + \dfrac{x}{\lambda + x} \right), \end{cases}$$

Aim:

## Theorem (DM, Y. Patsios)

*Given any smooth map*

$$F\colon [0,1] \to [0,1]$$

*there exists a "3D-variant" of the above predator-prey model for which a suitable (2D) first return map "mimics" the behaviour of the (1D) map $F$.*

## Theorem (DM, Y. Patsios)

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## Theorem (DM, Y. Patsios)

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*there exists a "3D-variant" of the above predator-prey model for which a suitable (2D) first return map "mimics" the behaviour of the (1D) map $P$.*

Remarks:

- This way of formulating a theorem is of course unacceptably vague
- First return maps are by definition diffeomorphisms whereas $F$ need not be !!!
- We would like to stay as close as possible to a realistic predator-prey model, but here we focused on ease of presentation
- The 3D-variant is easily constructed and the dynamic behaviour is easily verified with standard ode-solvers
- Ideally we think of the 2D diffeo $F$ as a map with a 1D attracting invariant curve $\gamma$ for which $F|_\gamma = P$, but this is in general too much to demand.

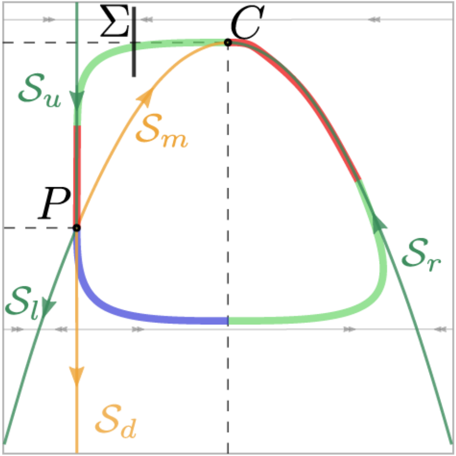
Set up:

$$\left\{\begin{array}{lcl}\dot{x} & = & \dots \\ \dot{y} & = & \dots \\ \dot{z} & = & \epsilon h(x,y,z,\epsilon)\end{array}\right. \quad \Bigg\} \text{ like before}$$

By adding a second slow variable ($\dot{z} = O(\epsilon)$ the critical curve, a parabola, becomes a critical surface.
We can then trace the evolution of $z$ by computing integrals over parts of the critical surface, which projects trivially on the former critical curve.

Taking a section $\Sigma = \{x = c\}$ between the fold and the plane we easily conclude the existence of a first return map

$$\mathcal{P} : \Sigma \to \Sigma$$



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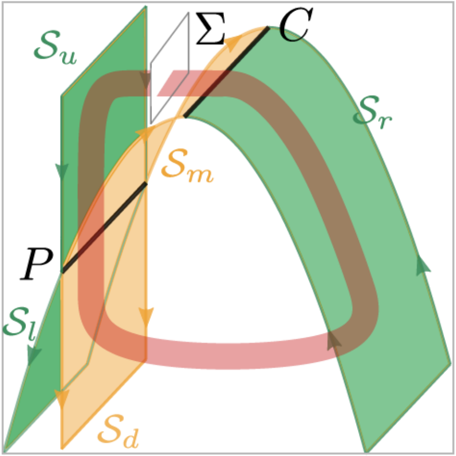

## The amended system

$$\begin{cases} \dot{x} &= x(1 + 2x - x^2 - y) \\ \dot{y} &= \epsilon y(x - \frac{1}{2}) \\ \dot{z} &= \epsilon h(x, y, z, \epsilon) \end{cases}$$

## The amended system

$$\begin{cases} \dot{x} &= x(1+2x-x^2-y)\Omega(y,z) \\ \dot{y} &= \epsilon y(x-\frac{1}{2}) \\ \dot{z} &= \epsilon h(x,y,z,\epsilon) \end{cases}$$

We assume $\Omega > 0$.

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**Slow-fast analysis**

The parabolic critical surface is given by

$$y = 1 + 2x - x^2$$

with a top line along $\{x = 1, y = 2\}$. Since $\dot{y}|_{x=1,y=2} = \epsilon > 0$, the dynamics point upwards near the fold so it is a jump situation

Slow dynamics on the parabolic surface:

$$\frac{dz}{dy} = \left. \frac{h(x, y, z, 0)}{(\frac{1}{2} - x)y} \right|_{y=1+2x-x^2}$$

We make it really easy for ourselves and assume

$$h = h_0(y) \times \left((\tfrac{1}{2} - x)y\right) + O(y - 1 - 2x + x^2)$$

Then the slow dynamics becomes trivial

$$\frac{dz}{dy} = h_0(y) \implies z_{jump} = z_0 + \int_{y_0}^{2} h_0(y)dy.$$

We will choose $h_0$ a bit later.

Slow dynamics on the invariant plane $\{x = 0\}$:

$$\begin{cases} y' &= -\frac{y}{2} \\ z' &= h(0, y, z, 0) \end{cases}$$

Here we make it ourselves really really easy by assuming

$$h(0, y, z, 0) = 0$$

So the dynamics is fully understood by the planar model.
The two conditions are compatible when $h_0(1) = 0$.

## Entry-exit mechanism

$$\begin{cases} \dot{x} &= x(1+2x-x^2-y)\Omega(y,z) \\ \dot{y} &= \epsilon y(x-\frac{1}{2}) \\ \dot{z} &= \epsilon h(x,y,z,\epsilon) \end{cases}$$

The divergence on $\{x=0\}$:

$$(1-y)\Omega(y,z)$$

So the divergence integral shows the exit point:

$$\int_{y_{entry}}^{y_{exit}} \frac{(1-y)\Omega(y,z)}{-y/2} dy = 0.$$

and

$$z_{exit} = z_{entry}$$

First return map (following “Chaotic attractors of relaxation oscillators”, Nonlinearity 2006, by Guckenheimer, Wechselberger and Lai-Sang Young)

$$(y_{entry}, z_{entry}) \mapsto (y_{jump}, z_{jump}) + o_\epsilon(1)$$

where

$$y_{jump} = 2$$

and $z_{jump}$ is implicitly defined by

$$\int_{y_{entry}}^{y_{exit}} \frac{(1-y)\Omega(y,z)}{-y/2} dy = 0$$

$$(y_{exit}, z_{exit}) = (y_0, z_0)$$

$$z_{jump} = z_0 + \int_{y_0}^{2} h_0(y) dy.$$

So after one iteration we assume $y_{entry} = y_{jump}$ and find a 1-D map

$$z_{entry} \mapsto z_{jump} + o(1)$$

where $z_{jump}$ is implicitly defined by

$$\int_2^{y_{exit}} \frac{(1-y)\Omega(y,z)}{-y/2} dy = 0$$
$$(y_{exit}, z_{entry}) = (y_0, z_0)$$
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## Eliminating variables

Suppose we want to reverse engineer a 1-D map $z \mapsto P(z)$.

$$\int_2^{y_{exit}} \frac{(1-y)\Omega(y,z)}{-y/2} dy = 0$$

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$$P(z_{entry}) = z_{entry} + \int_{y_{exit}}^{2} h_0(y) dy.$$

Next, choose

$$h_0(y) = 1 - y$$

so

$$y_{exit} = 1 - \sqrt{(y-1)^2 + 2P(z_{entry}) - 2z_{entry}}$$

This leads to a very nice implicit expression for the limiting 1-D map:

$$\int_2^{1-\sqrt{(y-1)^2+2P(z_{entry})-2z_{entry}}} \frac{(1-y)\Omega(y,z)}{-y/2} dy = 0$$

Reverse engineering $P$: consider this implicit expression as an equation for the unknown function $\Omega$.

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Reverse engineering $P$: consider this implicit expression as an equation for the unknown function $\Omega$.
Example solution:

$$\Omega(y,z) = y\left(1 + (y-1)^2 - (y-1)\bar{\Omega}(z)\right),$$

with

$$\bar{\Omega}(z) = \frac{3(2+P-z)(P-z)}{1+(1-2z+2P)^{3/2}}$$

We could now try to follow the techniques in
"Chaotic attractors of relaxation oscillators", Nonlinearity 2006, by
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However, in that paper conditions are quite strong and possibly require partly numerical verification!
We prefer to apply "Topological horseshoes", Trans. AMS 2001, by Kennedy and Yorke

# Kennedy-Yorke Topological horseshoes

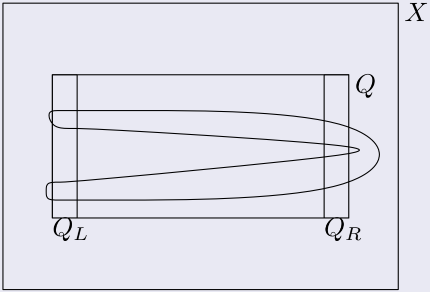
Let $Q_L, Q_R \subset Q \subset X \subset \mathbb{R}^n$ be compact sets, and assume $Q \subset X$ is connected. Let

$$f : Q \to X$$

be continuous.

Assume furthermore $Q_L \cap Q_R = \emptyset$. Finally assume the "crossing number" $M \geq 2$.

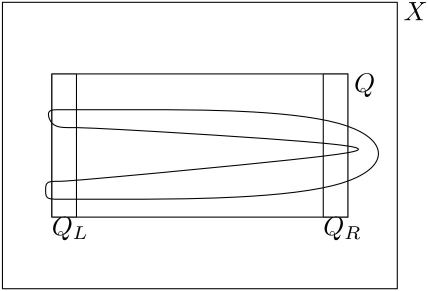
Then there is a closed invariant subset $C$ of $Q$ for which $f|_C$ is semi-conjugate to a one-sided shift on $M$ symbols.

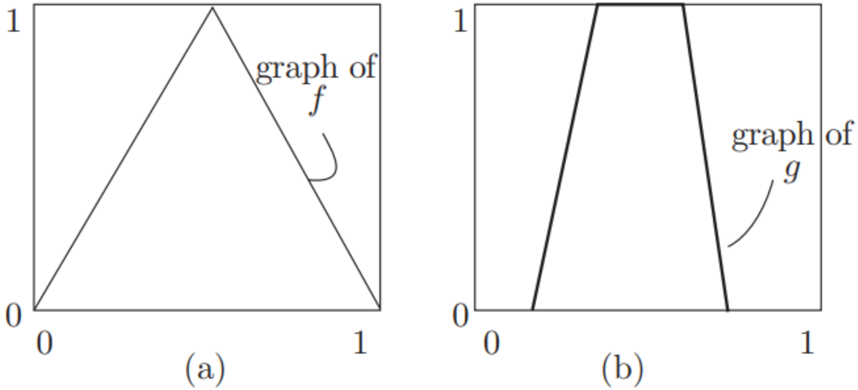


## Crossing number

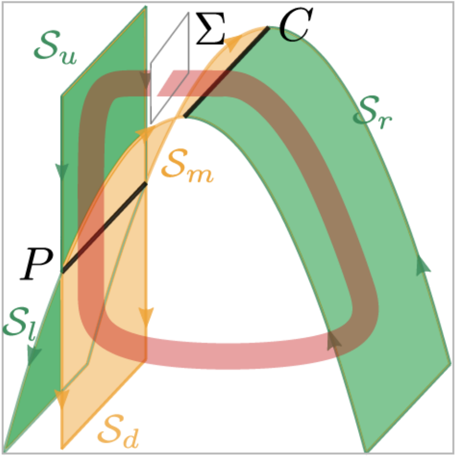
We define a *connection* as a compact connected subset of $Q$ intersecting with both $Q_L$ and $Q_R$

The crossing number $M$ is the largest number such that any connection contains at least $M$ *disjoint* compact connected subsets whose $f$-image is a connection





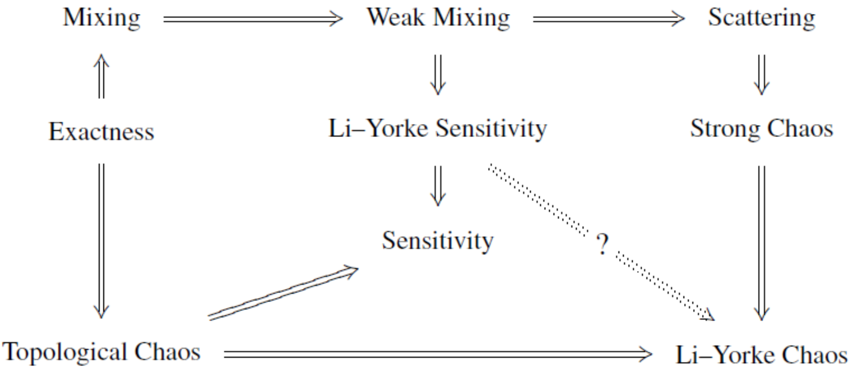

(b) satisfies conditions, (a) not.




### Theorem

*Let $P\colon [0,1] \to [0,1]$ satisfy the conditions of Kennedy-Yorke "in a stable way", then we can lift the one-dimensional sets $Q_L$, $Q_R$, $Q$ and $X$ to the plane so that the lifted sets satisfy the conditions of Kennedy-Yorke for the slow-fast return map $\mathcal{P}$*

The conclusion is the presence of chaos but not in a very strong sense...

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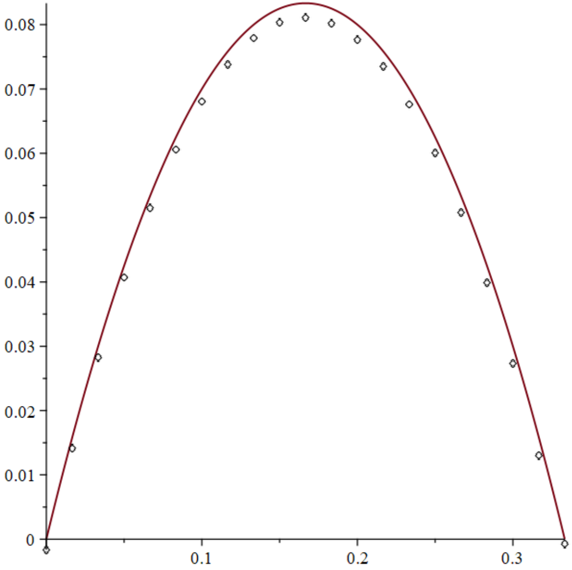
Conclusions:

- There is a rigorous proof of the existence of invariant sets with chaotic dynamics
- By making the invariant plane non-hyperbolic, the slow-fast analysis does not change at all and the construction is easily verified numerically!

$$\begin{cases} \dot{x} &= x^{2}(1+2x-x^2-y)\Omega(y,z) \\ \dot{y} &= \epsilon y(x-\frac{1}{2}) \\ \dot{z} &= \epsilon h(x,y,z,\epsilon) \end{cases}$$

- We didn't really prove that the invariant set is an attractor (project with X. Zhang)
- We didn't study how the invariant set behaves asymptotically as $\epsilon \to 0$ (same project)
- What about reverse engineering higher dimensional maps
- try the application of the more recent Lai-Sang conditions (Annals paper) to deal with chaos from a measure-theoretic point of view without resorting to numerics

## Numerical simulations with the non-hyperbolic model and $\epsilon = 0.0001$



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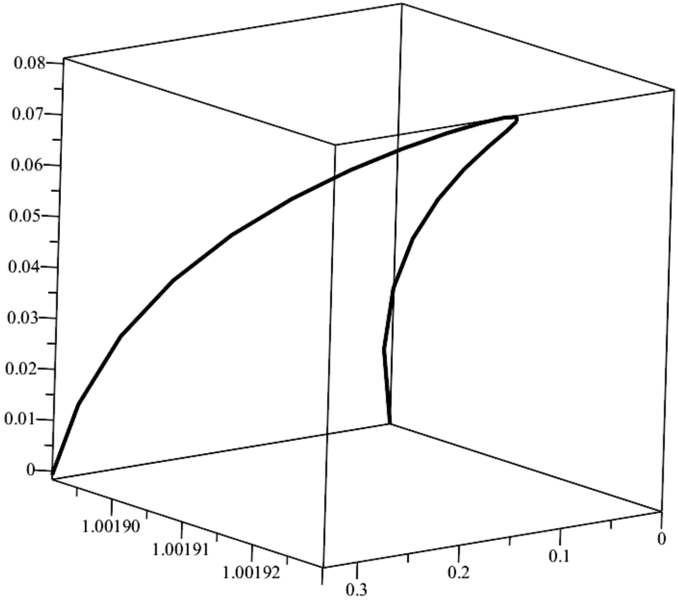

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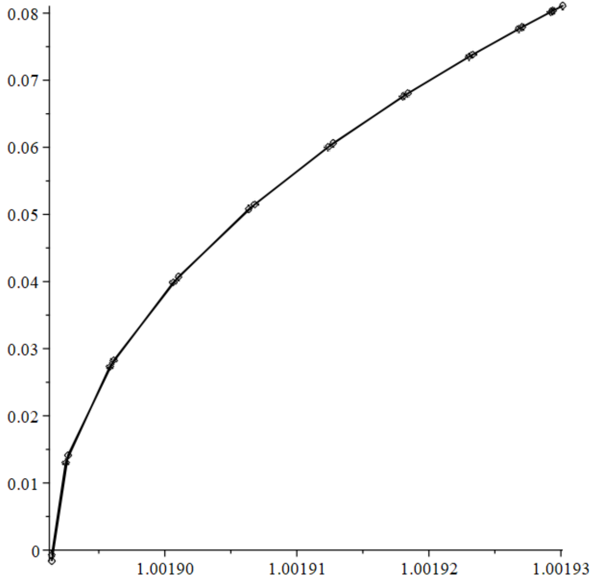

Thank you for your attention!