Structural stability in a class of refractive partially integrable vector fields

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$$Z(x, y, z) = \begin{cases} X(x, y, z), \; z \geq 0, \\ Y(x, y, z), \; z \leq 0, \end{cases} \tag{1}$$

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In other words, if $Z \in \mathfrak{R}$ then the restriction $Z_{|_{\mathbb{S}^2_\lambda}} \in \mathfrak{R}^{S_\lambda}$, for all $\lambda > 0$.

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- These results pave the way to prove Theorem C involving 3−dimensional refractive piecewise smooth vector fields in $\mathfrak{R}$.

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Ultimately, the tangency set of Z is $S_Z = S_X \cup S_Y$.

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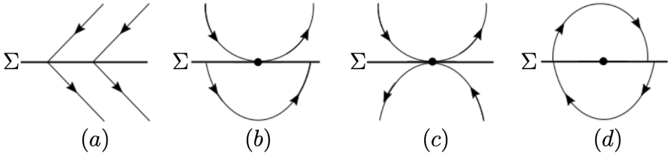

FIGURE 1. (a) Crossing region; (b) Parabolic fold–fold point; (c) Hyperbolic fold–fold point; (d) Elliptic fold–fold point.

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$\phi_X(\mathrm{x}) = -\mathrm{x} + \alpha_X \mathrm{x}^2 - \alpha_X^2 \mathrm{x}^3 + \mathcal{O}_4(\mathrm{x})$ and
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Which allows us to define a return map ϕ_Z by

$$\phi_Z(\mathrm{x}) = \phi_Y \circ \phi_X(\mathrm{x}) = \mathrm{x} + (\alpha_Y - \alpha_X)\mathrm{x}^2 + (\alpha_Y - \alpha_X)^2 \mathrm{x}^3 + \mathcal{O}_4(\mathrm{x}).$$

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The generic condition (G), impose that $\phi_X''(0) \neq \phi_Y''(0)$. This condition implies that none of the trajectories of Z, in a neighborhood of p, is a closed trajectory.

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Besides, we say that a closed poly-trajectory Γ of Z is generic (or elementary) if it is of the type I and its first return map $\pi : \Sigma \to \Sigma$ satisfies $\pi^{'}(p) \neq 1$, for all $p \in \Sigma \cap \Gamma$.

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We also say that Γ is a quasi-generic poly-trajectory of type I if $\pi : \Sigma \to \Sigma$ with $\pi^{'}(p) = 1$ and $\pi^{''}(q) \neq 0$, for all $p \in \Sigma \cap \Gamma$.

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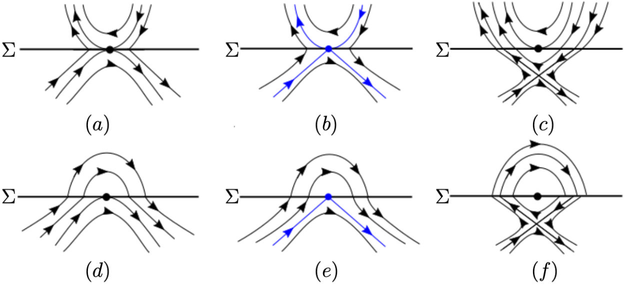

FIGURE 3. Fold–saddle points and their unfoldings.

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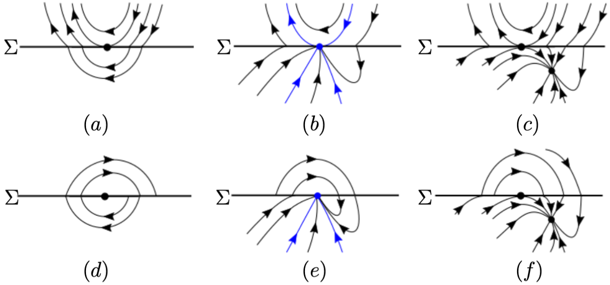

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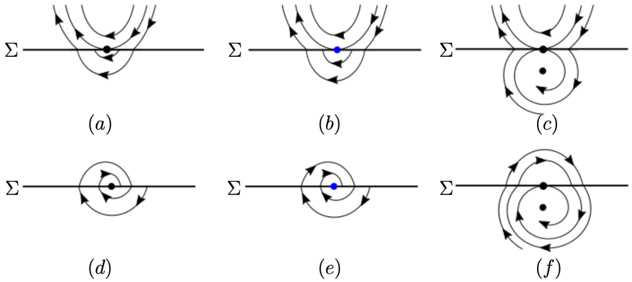

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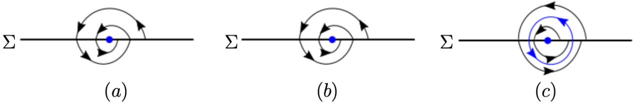

FIGURE 6. Quasi-generic elliptic fold–fold point and its unfolding, with $\beta < 0$.

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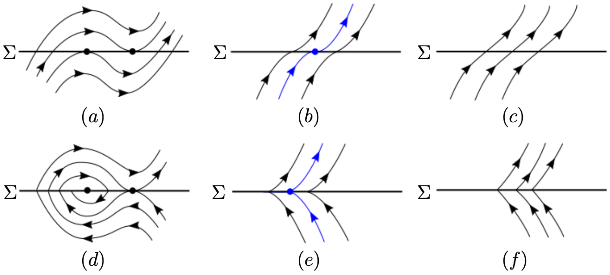

FIGURE 7. Cups-Cusp points and their unfolding.

Global structural stability on $\mathbb{S}^2_\lambda$.

Definition

$Z = (X, Y) \in \Sigma_0^{S_\lambda}$ if, and only if,

Global structural stability on $\mathbb{S}^2_\lambda$.

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(e) There is no separatrices connections.

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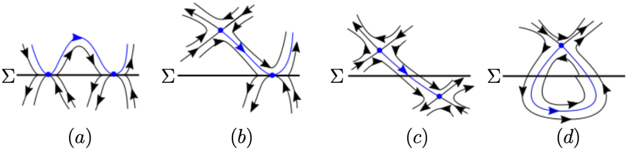

FIGURE 8. (a) Connection between two different fold–fold points; (b) saddle–fold connection; (c) saddle–saddle connection; (d) simple–loop.

Theorem A

There exists a subset $\Sigma_0^{S_\lambda} \subset \mathfrak{R}^{S_\lambda} \subset \mathcal{X}^{S_\lambda}$ *satisfying:*

(i) *It has a simple and comprehensive description.*

(ii) $Z \in \mathfrak{R}^{S_\lambda}$ *is structurally stable if, and only if,* $Z \in \Sigma_0^{S_\lambda}$.

(iii) $\Sigma_0^{S_\lambda}$ *is open and dense in* $\mathfrak{R}^{S_\lambda}$.

The generic bifurcation manifold on $\mathbb{S}^2_\lambda$

Consider the bifurcation set $\mathfrak{R}_1^{S_\lambda} = \mathfrak{R}^{S_\lambda} \setminus \Sigma_0^{S_\lambda}$.

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$$\Sigma_1^{S_\lambda} = \Sigma_1^{S_\lambda}(a_1) \cup \Sigma_1^{S_\lambda}(a_2) \cup \Sigma_1^{S_\lambda}(b_1) \cup \Sigma_1^{S_\lambda}(b_2) \cup \Sigma_1^{S_\lambda}(c_1) \cup \Sigma_1^{S_\lambda}(c_2) \cup \Sigma_1^{S_\lambda}(d_1) \cup \Sigma_1^{S_\lambda}(d_2) \cup \Sigma_1^{S_\lambda}(e),$$

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where,

- $Z = (X, Y) \in \Sigma_1^{S_\lambda}(a_1)$ if all equilibrium points of X and Y are hyperbolic except one of them that is either a saddle-node or a Hopf equilibrium point. All of them are away from Σ. Moreover, the conditions (b), (c), (d), (e) and (f) of Definition of $\Sigma_0^{S_\lambda}$ are satisfied.

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where,

- $Z = (X, Y) \in \Sigma_1^{S_\lambda}(a_1)$ if all equilibrium points of X and Y are hyperbolic except one of them that is either a saddle-node or a Hopf equilibrium point. All of them are away from Σ. Moreover, the conditions (b), (c), (d), (e) and (f) of Definition of $\Sigma_0^{S_\lambda}$ are satisfied.
- $Z = (X, Y) \in \Sigma_1^{S_\lambda}(a_2)$ if Z has only one equilibrium–fold (or fold–equilibrium) point $p \in \Sigma$. In addition we consider the non-degeneracy conditions given previously. Moreover, the conditions (b), (c), (d), (e) and (f) of Definition of $\Sigma_0^{S_\lambda}$ are satisfied.

The generic bifurcation manifold on $\mathbb{S}^2_\lambda$

- $Z = (X, Y) \in \Sigma_1^{S_\lambda}(b_1)$ if all periodic orbits of X and Y are hyperbolic except one of them which is of saddle-node type. None of them is tangent to Σ. Moreover, the conditions (a), (c), (d), (e) and (f) of Definition of $\Sigma_0^{S_\lambda}$ are satisfied.

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- $Z = (X, Y) \in \Sigma_1^{S_\lambda}(b_2)$ if all periodic orbits are hyperbolic and just one of them is generically tangent to Σ. Moreover, the conditions (a), (c), (d), (e) and (f) of Definition of $\Sigma_0^{S_\lambda}$ are satisfied.

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- $Z = (X, Y) \in \Sigma_1^{S_\lambda}(c_1)$ if all poly-trajectories of Z are elementary except one, Γ, of type I such that $\pi^{'}(q) = 1$ and $\pi^{''}(q) = d^{''}(q) \neq 0$, for all $q \in \Sigma \cap \Gamma$. Moreover, the conditions (a), (b), (d), (e) and (f) of Definition of $\Sigma_0^{S_\lambda}$ are satisfied.

The generic bifurcation manifold on $\mathbb{S}^2_\lambda$

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- $Z = (X, Y) \in \Sigma_1^{S_\lambda}(c_2)$ if all poly-trajectories of Z are elementary except one of them which is of type II with just one hyperbolic fold–fold point. Moreover, the conditions (a), (b), (d), (e) and (f) of Definition of $\Sigma_0^{S_\lambda}$ are satisfied.

- $Z = (X, Y) \in \Sigma_1^{S_\lambda}(d_1)$ if all tangency points are generic fold–fold points except one of them which is a quasi-generic fold–fold point. Moreover, the conditions (a), (b), (c), (e) and (f) of Definition of $\Sigma_0^{S_\lambda}$ are satisfied.

- $Z = (X, Y) \in \Sigma_1^{S_\lambda}(d_1)$ if all tangency points are generic fold–fold points except one of them which is a quasi-generic fold–fold point. Moreover, the conditions (a), (b), (c), (e) and (f) of Definition of $\Sigma_0^{S_\lambda}$ are satisfied.
- $Z = (X, Y) \in \Sigma_1^{S_\lambda}(d_2)$ if all tangency points are generic fold–fold points except one of them which is a simple (or generic) cusp–cusp. Moreover, the conditions (a), (b), (c), (e) and (f) of Definition of $\Sigma_0^{S_\lambda}$ are satisfied.

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- $Z = (X, Y) \in \Sigma_1^{S_\lambda}(e)$ if there is just one separatrix connection which is $quasi - generic$. Moreover, the conditions (a), (b), (c), (d) and (f) of Definition of $\Sigma_0^{S_\lambda}$ are satisfied.

The generic bifurcation manifold on $\mathbb{S}^2_\lambda$

Theorem B

There exists a immersed codimension-one submanifold $\Sigma_1^{S_\lambda} \subset \mathfrak{R}^{S_\lambda}$ satisfying:

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(ii) *For any* $Z_0 \in \Sigma_1^{S_\lambda}$, *there exists a neighborhood* $\mathcal{B}(Z_0) \subset \Sigma_1^{S_\lambda}$ *such that any* $Z \in \mathcal{B}$ *is* Σ*-equivalent to* Z_0 *(in the intrinsic topology of* $\mathfrak{R}_1$*). Thus,* $\Sigma_1^{S_\lambda}$ *is open in* $\mathfrak{R}_1$ *with the intrinsic topology.*

The generic bifurcation manifold on $\mathbb{S}^2_\lambda$

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(iii) $\Sigma_1^{S_\lambda}$ *is dense in* $\mathfrak{R}_1$.

The generic bifurcation manifold on $\mathbb{S}^2_\lambda$

Theorem B

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(i) $\Sigma_1^{S_\lambda}$ *is characterized.*

(ii) *For any $Z_0 \in \Sigma_1^{S_\lambda}$, there exists a neighborhood $\mathcal{B}(Z_0) \subset \Sigma_1^{S_\lambda}$ such that any $Z \in \mathcal{B}$ is Σ-equivalent to Z_0 (in the intrinsic topology of $\mathfrak{R}_1$). Thus, $\Sigma_1^{S_\lambda}$ is open in $\mathfrak{R}_1$ with the intrinsic topology.*

(iii) $\Sigma_1^{S_\lambda}$ *is dense in $\mathfrak{R}_1$.*

Remark

It is important to note that we use the intrinsic topology in item (ii) of Theorem B because it is finer (i.e. it has more open sets) than the ambient topology.

We use spherical coordinates to consider $Z = (X, Y) \in \mathfrak{R}$ as a 1$-$parameter family of refractive piecewise smooth vector fields $Z_\lambda \in \mathfrak{R}^{S_\lambda}$.

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The spherical coordinates on $\mathbb{R}^3$ is given by
$x = \rho \cos\theta \sin\phi,\ \ y = \rho \sin\theta \sin\phi,\ \ z = \rho \cos\phi,$

Stability conditions in $\mathfrak{R}$

We use spherical coordinates to consider $Z = (X, Y) \in \mathfrak{R}$ as a $1-$parameter family of refractive piecewise smooth vector fields $Z_\lambda \in \mathfrak{R}^{S_\lambda}$.

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$X(\rho,\theta,\phi) = (R_1(\rho,\theta,\phi), \Theta_1(\rho,\theta,\phi), \Phi_1(\rho,\theta,\phi)),$
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$Y(\rho,\theta,\phi) = (R_2(\rho,\theta,\phi), \Theta_2(\rho,\theta,\phi), \Phi_2(\rho,\theta,\phi))$,and the discontinuity set $\Sigma = \{(\rho,\theta,\phi); \phi = \pi/2\}$. So, the refractive vector field $Z : U \subset \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$Z(\rho,\theta,\phi) = \begin{cases} X(\rho,\theta,\phi),\ 0 \leq \phi \leq \pi/2, \\ Y(\rho,\theta,\phi),\ \pi/2 \leq \phi \leq \pi. \end{cases} \qquad (2)$$

Stability conditions in $\mathfrak{R}$

Note that if all the spheres are invariant by the flow of Z, i.e., if $Z \in \mathfrak{R}$, then $R_i(\rho, \theta, \phi) \equiv 0$, $i = 1, 2$.

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We can define a $1-$parameter family of refractive vector fields in $\mathfrak{R}^{S_\lambda}$, $Z_\mu : I \times \mathbb{S}^2_\lambda \to T\mathbb{S}^2_\lambda$, writing

$$Z_\mu(\mu, \theta, \phi) = (X_\mu(\mu, \theta, \phi), Y_\mu(\mu, \theta, \phi)), \tag{3}$$

with $Z_\mu \in \mathfrak{R}^{S_\lambda}$ for all $\mu \in I$.

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Thus, it is possible to make a correspondence between a $1-$parameter family $Z_\mu \in \mathfrak{R}^{S_\lambda}$, with $\mu \in I$, and a unique $Z : V_{\lambda_0} \to \mathbb{R}^3 \in \mathfrak{R}$,

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Thus, it is possible to make a correspondence between a $1-$parameter family $Z_\mu \in \mathfrak{R}^{S_\lambda}$, with $\mu \in I$, and a unique $Z : V_{\lambda_0} \to \mathbb{R}^3 \in \mathfrak{R}$, where V_{λ_0} is a tubular neighborhood of $\mathbb{S}^2_{\lambda_0}$ given by $V_{\lambda_0} = \{(\mu, \theta, \phi) \in \mathbb{R}^3; \mu \in I,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \pi\}$. To do that, for each $Z_\mu(\mu, \theta, \phi) = (X_\mu(\mu, \theta, \phi), Y_\mu(\mu, \theta, \phi))$ defined by (3) we associate $Z(\mu, \theta, \phi) = (X(\mu, \theta, \phi), Y(\mu, \theta, \phi))$ defined by (2).

On the following we consider $Z = (X, Y) \in \mathfrak{R}$ and the restriction $Z_\lambda = Z_{|_{\mathbb{S}^2_\lambda}}$ to the sphere $\mathbb{S}^2_\lambda$.

Generic set

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(a) $Z_\lambda \in \Sigma_0^{S_\lambda}$ for all $\lambda \in V_{\lambda_0}$;

(b) $Z_{\lambda_0} \in \Sigma_1^{S_\lambda}$ and $Z_\lambda \in \Sigma_0^{S_\lambda}$ for all $\lambda \in V_{\lambda_0} \setminus \{\lambda_0\}$;

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(a) $Z_{\lambda_0} \in \Sigma_1^{S_\lambda}(b_1) \cup \Sigma_1^{S_\lambda}(b_2)$, there exist a sequence $(\lambda_n)_{n\in\mathbb{N}} \subset V_{\lambda_0}$ such that $\lambda_n \to \lambda_0$ with $\lambda_n \neq \lambda_0$ and $Z_{\lambda_n} \in \Sigma_1^{S_\lambda}(ss) \cup \Sigma_1^{S_\lambda}(fs) \cup \Sigma_1^{S_\lambda}(ff)$ for all $n \in \mathbb{N}$, and $Z_\lambda \in \Sigma_0^{S_\lambda}$ for all $\lambda \in V_{\lambda_0} \setminus \{\lambda_0, \ \lambda_1, \ldots\}$;

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Definition

We say that $Z \in \Sigma_0 \subset \mathfrak{R}$ if, and only if, for each λ_0 there exists a neighborhood V_{λ_0} of λ_0 in which one of the following conditions is satisfied:

(a) $Z_{\lambda_0} \in \Sigma_1^{S_\lambda}(b_1) \cup \Sigma_1^{S_\lambda}(b_2)$, there exist a sequence $(\lambda_n)_{n\in\mathbb{N}} \subset V_{\lambda_0}$ such that $\lambda_n \to \lambda_0$ with $\lambda_n \neq \lambda_0$ and $Z_{\lambda_n} \in \Sigma_1^{S_\lambda}(ss) \cup \Sigma_1^{S_\lambda}(fs) \cup \Sigma_1^{S_\lambda}(ff)$ for all $n \in \mathbb{N}$, and $Z_\lambda \in \Sigma_0^{S_\lambda}$ for all $\lambda \in V_{\lambda_0} \setminus \{\lambda_0,\ \lambda_1, \ldots\}$;

(b) $Z_{\lambda_0} \in \Sigma_1^{S_\lambda}(c_1) \cup \Sigma_1^{S_\lambda}(c_2)$, there exist a sequence $(\lambda_n)_{n\in\mathbb{N}} \subset V_{\lambda_0}$ such that $\lambda_n \to \lambda_0$ with $\lambda_n \neq \lambda_0$ and $Z_{\lambda_n} \in \Sigma_1^{S_\lambda}(ss) \cup \Sigma_1^{S_\lambda}(fs) \cup \Sigma_1^{S_\lambda}(ff)$ for all $n \in \mathbb{N}$, and $Z_\lambda \in \Sigma_0^{S_\lambda}$ for all $\lambda \in V_{\lambda_0} \setminus \{\lambda_0,\ \lambda_1, \ldots\}$;

Generic set

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(c) $Z_{\lambda_0} \in \Sigma_1^{S_\lambda}(sl)$, there exist a sequence $(\lambda_n)_{n\in\mathbb{N}} \subset V_{\lambda_0}$ such that $\lambda_n \to \lambda_0$ with $\lambda_n \neq \lambda_0$ and $Z_{\lambda_n} \in \Sigma_1^{S_\lambda}(ss) \cup \Sigma_1^{S_\lambda}(fs)$ for all $n \in \mathbb{N}$, and $Z_\lambda \in \Sigma_0^{S_\lambda}$ for all $\lambda \in V_{\lambda_0} \setminus \{\lambda_0,\ \lambda_1, \ldots\}$.

Main Theorem

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If $Z \in \mathfrak{R}$ *is structurally stable, then* $Z \in \Sigma_0$.

Idea of the proof

Let $Z \in \mathfrak{R} \setminus \Sigma_0$.

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As $Z_{\lambda_0} \notin \Sigma_0^{S_\lambda} \cup \Sigma_1^{S_\lambda}$ and $\Sigma_0^{S_\lambda}$ is dense in $\mathfrak{R}^{S_{\lambda_0}}$, there exist $\widetilde{Z}_{\lambda_0} \in \mathcal{B} \cap \Sigma_0^{S_\lambda}$,

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As $Z_{\lambda_0} \notin \Sigma_0^{S_\lambda} \cup \Sigma_1^{S_\lambda}$ and $\Sigma_0^{S_\lambda}$ is dense in $\mathfrak{R}^{S_{\lambda_0}}$, there exist $\widetilde{Z}_{\lambda_0} \in \mathcal{B} \cap \Sigma_0^{S_\lambda}$,which can be extended to a refractive piecewise vector field $\widetilde{Z} \in \mathcal{U} \subset \mathfrak{R}$.

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As $Z_{\lambda_0} \notin \Sigma_0^{S_\lambda} \cup \Sigma_1^{S_\lambda}$ and $\Sigma_0^{S_\lambda}$ is dense in $\mathfrak{R}^{S_{\lambda_0}}$, there exist $\widetilde{Z}_{\lambda_0} \in \mathcal{B} \cap \Sigma_0^{S_\lambda}$,which can be extended to a refractive piecewise vector field $\widetilde{Z} \in \mathcal{U} \subset \mathfrak{R}$.

As $\widetilde{Z}_{\lambda_0}$ is structurally stable in $\mathfrak{R}^{S_{\lambda_0}}$, there exist $\varepsilon_0 > 0$ and a tubular neighborhood $V_{\lambda_0} = \{(\lambda, \theta, \phi) \in \mathbb{R}^3; \lambda_0 - \varepsilon_0 \leq \lambda \leq \lambda_0 + \varepsilon_0,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \pi\}$ of $\mathbb{S}^2_{\lambda_0}$ such that $\widetilde{Z}_\lambda$ is Σ-equivalent to $\widetilde{Z}_{\lambda_0}$ for all $\lambda \in (\lambda_0 - \varepsilon_0, \lambda_0 + \varepsilon_0)$.

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So, we have proved that if Z is such that $Z_{\lambda_0} \notin \Sigma_0^{S_\lambda} \cup \Sigma_1^{S_\lambda}$, given any neighborhood $\mathcal{U} \subset \mathfrak{R}$ of Z there exists $\widetilde{Z} \in \mathcal{U}$ such that $\widetilde{Z}$ is not Σ-equivalent to Z.

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This implies that Z is not structurally stable.

Thank you for your attention!

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