LIMIT CYCLES OF DISCONTINUOUS PIECEWISE LINEAR DIFFERENTIAL SYSTEMS WITH TWO ZONES SEPARATED BY A PARABOLA

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Abstract. We prove that if the unperturbed linear center $\dot{x} = y$, $\dot{y} = -x$ is at the vertex of the parabola $y = x^2$, then perturbing this center by a discontinuous piecewise linear differential system with two zones separated by the parabola the perturbed system can exhibit 3 limit cycles. We note that if we perturb the same linear center by a discontinuous piecewise linear differential system with two zones separated by the straight line $y = 0$ the perturbed system can exhibit at most 2 limit cycles.

1. Introduction and statement of the main results

The study of piecewise linear differential systems is relatively recent. The contributions of Andronov, Vitt and Khaikin [1] provided the basis for the development of the theory for these systems, which has received much attention from researchers. One of the reasons for this interest in the mathematical community is that these systems can be used to model applied problems, such as electronic circuits, biological systems, mechanical devices, etc, see for instance the book [3]. Thus, in recent years, the theory of piecewise linear differential systems has been increasingly developed and studied in order to understand the dynamics that such systems may have. In this sense one of the points of greatest interest is to obtain a lower bound for the maximum number of limit cycles that may arise around a single equilibrium point on the discontinuity set (i.e., on the region separating the linear differential systems). Remember that a *limit cycle* of a differential system is a periodic orbit which is isolated in the set of all periodic orbits of the system.

This investigation started with the simplest possible case: the continuous piecewise linear differential systems with two zones separated by a straight line. Lum and Chua [22] conjectured that the maximum number of limit cycles that can arise in such systems is one. Later this conjecture was proved by Freire, Ponce, Rodrigo and Torres [9] and more recently received an easier proof in [18]. After the closure of this case the attention turned to the class of piecewise linear differential systems with two zones, still separated by a straight line, but without the assumption of continuity. Several authors has been investigating the limit cycles for this class of systems, see for instance the articles [2, 4, 5, 7, 10, 11, 12, 13, 14, 15, 19, 20, 23] and found that the lower bounds for the maximum number of limit cycles of discontinuous piecewise linear differential system with two zones separated by a straight

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line can be three. But in the case that we perturb the linear center $\dot{x} = y$, $\dot{y} = -x$ inside the class of piecewise linear differential systems with two pieces separated by the straight line $y = 0$ it is known that the maximum number of limit cycles which can appear for small perturbations is two, see [17]. In order to extend this research, we will consider discontinuous piecewise linear differential system with two zones, but separated by a parabola instead of a straight line.

Let D be an open subset of $\mathbb{R}^2$ and consider a smooth function $h : D \subset \mathbb{R}^2 \to \mathbb{R}$. Suppose that $0 \in \mathbb{R}$ is a regular value of h and D is an open neighborhood of the origin, so that the set $\Sigma = h^{-1}(0) \subset D$ is a differentiable curve in the plane. The curve Σ, called *discontinuity set*, separates D in two open regions $S_1 = \{(x, y) \in D : h(x, y) > 0\}$ and $S_2 = \{(x, y) \in D : h(x, y) < 0\}$. We define a *planar discontinuous piecewise linear differential system*, and denote it by $X = (X^1, X^2)$, as

$$X(x,y) = \begin{cases} X^1(x,y) & \text{if} \quad (x,y) \in \bar{S}_1, \\ X^2(x,y) & \text{if} \quad (x,y) \in \bar{S}_2, \end{cases} \tag{1}$$

where $X^1, X^2 : D \subset \mathbb{R}^2 \to \mathbb{R}^2$ are planar linear differential systems.

In accordance with Filippov [8] we distinguish the following open regions in the discontinuity set Σ:

(1) *Crossing region*: $\Sigma_c = \{p \in \Sigma : X^1h(p).X^2h(p) > 0\}$,
(2) *Sliding region*: $\Sigma_s = \{p \in \Sigma : X^1h(p).X^2h(p) < 0\}$,

where $X^jh(p) = \big\langle \nabla h(p), X^j(p) \big\rangle$ for $j = 1, 2$.

In this work we consider a planar discontinuous piecewise linear differential system with two zones, S_1 and S_2, separated by a parabola. We assume without loss of generality that the function $h(x, y) = y - x^2$ defines Σ. Considering a subset $B \subset D$, let $\chi_B(t, x)$ be the *characteristic function* defined as

$$\chi_B(x,y) = \begin{cases} 1 & \text{if} \quad (x,y) \in B, \\ 0 & \text{if} \quad (x,y) \notin B. \end{cases}$$

So system (1) can be written as

$$X(x,y) = \sum_{j=1}^{2} \chi_{\bar{S}_j}(x,y) X^j(x,y). \tag{2}$$

Consider a linear perturbation of system (2) with a linear center at origin

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} y + \varepsilon(a_0 + a_1 x + a_2 y) \\ -x + \varepsilon(d_0 + d_1 x + d_2 y) \end{pmatrix} & \text{if} \quad h(x,y) > 0, \\ \\ \begin{pmatrix} y + \varepsilon(\alpha_0 + \alpha_1 x + \alpha_2 y) \\ -x + \varepsilon(\delta_0 + \delta_1 x + \delta_2 y) \end{pmatrix} & \text{if} \quad h(x,y) < 0. \end{cases} \tag{3}$$

Our main result on the limit cycles of system (3) is the following.

Theorem 1. *For $|\varepsilon| \neq 0$ sufficiently small and a discontinuity set Σ given by the parabola $h(x, y) = y - x^2 = 0$ the piecewise linear differential system* (3) *can have 3 limit cycles.*

We note that if the discontinuity set is the straight line $y = 0$, then the piecewise linear differential system (3) has at most two limit cycles for $|\varepsilon| \neq 0$ sufficiently small. This result has been proved in [17]. So when the discontinuity set becomes a parabola Theorem 1 shows that the maximum number of limit cycles that the piecewise linear differential system (3) can exhibit is greater than two.

2. Averaging theory for discontinuous piecewise differential systems

In this section we present the averaging theory for planar discontinuous piecewise differential systems (PDPDS). For more details, see [16].

Consider the following discontinuous piecewise differential system

$$x'(t) = \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x) + \varepsilon^3 R(t,x,\varepsilon), \tag{4}$$

with

$$\begin{aligned} F_i(t,x) = &\sum_{j=1}^{2} \chi_{\bar{S}_j}(t,x) F_i^j(t,x), \quad \text{for} \quad i=1,2, \quad \text{and} \\ R(t,x,\varepsilon) = &\sum_{j=1}^{2} \chi_{\bar{S}_j}(t,x) R^j(t,x), \end{aligned}$$

where $F_i^j : \mathbb{S}^1 \times D \to \mathbb{R}^2$, $R^j : \mathbb{S}^1 \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^2$ for $i=1,2$ and $j=1,2$ are continuous functions, T-periodic in the variable t and D is an open subset of $\mathbb{R}^2$. For $i=1,2$ denote

$$D_x F_i(t,z) = \sum_{j=1}^{2} \chi_{\bar{S}_j}(t,z) D_x F_i^j(t,z),$$

and define the averaged functions $f_1, f_2 : D \to \mathbb{R}^2$ as

$$\begin{aligned} f_1(z) = &\int_0^T F_1(t,z)\, dt, \quad \text{and} \\ f_2(z) = &\int_0^T \left(D_x F_1(t,z) y_1(t,z) + F_2(t,z)\right) dt, \end{aligned}$$

where

$$y_1(t,z) = \int_0^t F_1(s,z)\, ds.$$

For a proof of the following result see [16].

Theorem 2. *(The first order averaging theorem for PDPDS). Assume the following conditions.*

(H1) *For $i=1,2$ and $j=1,2$, the continuous functions F_i^j and R_i^j are locally Lipschitz with respect to x, and T-periodic with respect to the time t. Furthermore for $j=1,2$ the boundaries of S_j are piecewise C^k embedded hypersurfaces with $k \geq 1$.*

(H2) *There exists an open bounded set $C \subset D$ such that, for $|\varepsilon| \neq 0$ sufficiently small, every solution of system (4) starting in $\overline{C}$ reaches the set of discontinuity Σ only at its crossing region.*

(H3) *For $a^* \in C$ with $f_1(a^*) = 0$, there exists a neighborhood $U \subset C$ of a^* such that $f_1(z) \neq 0$ for all $z \in \overline{U} \setminus \{a^*\}$ and $d_B(f_1, U, 0) \neq 0$. ($d_B(f_1, U, 0)$ denotes the Brouwer degree of f_1 at 0).*

Then for $|\varepsilon| \neq 0$ sufficiently small, there exists a T-periodic solution $x(t,\varepsilon)$ of system (4) *such that $x(0,\varepsilon) \to a^*$ as $\varepsilon \to 0$.*

Remark 1. *If the determinant of Jacobian matrices $Df_1(a^*)$ in Theorem 2 is nonzero, then the Brouwer degree $d_B(f_1,U,0)$ is not zero, for more details see* [21].

3. Proof of the main results

In order to prove Theorem 1 we need the following lemma proved in Lemma 4.5 of [6].

Lemma 3. *Consider $p+1$ linearly independent functions $f_i : U \subset \mathbb{R} \to \mathbb{R}$, $i = 0,1,\dots,p$*

(i) *Given p arbitrary values $x_i \in U$, $i = 0,1,\dots,p$ there exist $p+1$ constants C_i, $i = 0,1,\dots,p$ such that*

$$f(x) := \sum_{i=0}^{p} C_i f_i(x) \tag{5}$$

is not the zero function and $f(x_i) = 0$ for $i = 0,1,\dots,p$.

(ii) *Furthermore if all f_i are analytical functions on U and there exists $j \in \{0,1,\dots,p\}$ such that $f_j|_U$ has constant sign, it is possible to get an f given by* (5)*, such that it has at least p simple zeros in U.*

Now consider the functions

$$\begin{aligned} k_0(r) &= \sqrt{-1+\sqrt{1+4r^2}},\\ k_1(r) &= \sqrt{-1+\sqrt{1+4r^2}+r^2\left(-3+\sqrt{1+4r^2}\right)},\\ k_2(r) &= r^2,\\ k_3(r) &= r^2\csc^{-1}\left(\frac{\sqrt{2}r}{\sqrt{-1+\sqrt{1+4r^2}}}\right), \end{aligned}$$

and define the set of functions $K = \{k_0,k_1,k_2,k_3\}$.

Lemma 4. *The functions of K are linearly independent on the interval $(0,\infty)$.*

Proof. To prove the assertion it is necessary and sufficient to show that $W_K(r) = W(k_0,k_1,k_2,k_3)(r) \neq 0$ on $(0,\infty)$, where $W(f_0,f_1,\dots,f_n)(t)$ denotes the Wronskian of the functions $f_0,f_1,\dots,f_n$ with respect to t, that is

$$W(f_0,f_1,\dots,f_n)(t) = \begin{vmatrix} f_0(t) & \cdots & f_n(t)\\ f_0'(t) & \cdots & f_n'(t)\\ \vdots & \ddots & \vdots\\ f_0^{(n)}(t) & \cdots & f_n^{(n)}(t) \end{vmatrix}.$$

So, using some algebraic manipulator, as Mathematica or Maple, we obtain

$$W_K(r) = \frac{16(1+4r^2+\sqrt{1+4r^2})P_1(r)}{(1+4r^2)^{7/2}},$$

where

$$P_1(r) \;=\; \sqrt{(2+6r^2+2r^4)\sqrt{1+4r^2}-(2+10r^2+10r^4)}.$$

It is easy to see that $P_1(r) \neq 0$, and $1+4r^2+\sqrt{1+4r^2} > 0$. Therefore $W_K(r) \neq 0$ on $(0,+\infty)$. This conclude the proof of the lemma. □

Proof of Theorem 2. Consider system (3). Using polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ system (3) becomes

$$(\dot r, \dot\theta) = (0,-1) + \varepsilon M(\theta, r), \tag{6}$$

where:

$$M(r,\theta) = \begin{cases} (A(\theta,r), B(\theta,r)) & \text{if } \sin\theta - r\cos^2\theta > 0, \\ (C(\theta,r), D(\theta,r)) & \text{if } \sin\theta - r\cos^2\theta < 0, \end{cases}$$

and

$$\begin{aligned} A(\theta,r) &= (a_0 + r(a_2+d_1)\sin\theta)\cos\theta + a_1 r\cos^2\theta + (d_0 + d_2 r\sin\theta)\sin\theta, \\ B(\theta,r) &= \frac{(d_0 + d_1 r\cos\theta + d_2 r\sin\theta)\cos\theta - (a_0 + a_1 r\cos\theta + a_2 r\sin\theta)\sin\theta}{r}, \\ C(\theta,r) &= (\alpha_0 + r(\alpha_2+\delta_1)\sin\theta)\cos\theta + \alpha_1 r\cos^2\theta + (\delta_0 + \delta_2 r\sin\theta)\sin\theta, \\ D(\theta,r) &= \frac{(\delta_0 + \delta_1 r\cos\theta + \delta_2 r\sin\theta)\cos\theta - (\alpha_0 + \alpha_1 r\cos\theta + \alpha_2 r\sin\theta)\sin\theta}{r}. \end{aligned}$$

Taking θ as the new time system (6) writes

$$\frac{dr}{d\theta} = \begin{cases} \dfrac{\varepsilon A(\theta,r)}{-1+\varepsilon B(\theta,r)} & \text{if } \sin\theta - r\cos^2\theta > 0, \\[2ex] \dfrac{\varepsilon C(\theta,r)}{-1+\varepsilon D(\theta,r)} & \text{if } \sin\theta - r\cos^2\theta < 0. \end{cases}$$

So system (6) and consequently system (3) become equivalent to

$$r' = \mathcal{R}(\theta, r, \varepsilon), \tag{7}$$

where the prime denotes derivatives with respect to the independent variable θ.

Expanding (7) in Taylor series at $\varepsilon = 0$ up to order 1 in ε we get

$$r' = \begin{cases} \varepsilon F_1(\theta,r) + \mathcal{O}(\varepsilon^2) & \text{if } \sin\theta - r\cos^2\theta > 0, \\ \varepsilon G_1(\theta,r) + \mathcal{O}(\varepsilon^2) & \text{if } \sin\theta - r\cos^2\theta < 0, \end{cases} \tag{8}$$

where

$$\begin{aligned} F_1(\theta,r) &= -a_1 r\cos^2\theta - (a_0 + (a_2+d_1) r\sin\theta)\cos\theta - (d_0 + d_2 r\sin\theta)\sin\theta, \\ G_1(\theta,r) &= -\alpha_1 r\cos^2\theta - (\alpha_0 + (\alpha_2+\delta_1) r\sin\theta)\cos\theta - (\delta_0 + \delta_2 r\sin\theta)\sin\theta. \end{aligned}$$

Clearly hypotesis (H1) of Theorem 2 holds for system (8). Furthermore, given

$$\theta_1(r) = \arccos\left(\sqrt{\frac{\sqrt{4r^2+1}-1}{2r^2}}\right) \quad \text{and} \quad \theta_2(r) = \arccos\left(-\sqrt{\frac{\sqrt{4r^2+1}-1}{2r^2}}\right),$$

we have that for $r > 0$, $\sin\theta - r\cos^2\theta > 0$ if and only if $\theta_1(r) < \theta < \theta_2(r)$; and $\sin\theta - r\cos^2\theta < 0$ if and only if $\theta_2(r) < \theta < \theta_1(r) + 2\pi$. Let $\hat h(\theta, r) = \sin\theta - r\cos^2\theta$, thus the set of discontinuity of system (8) is given by $\hat\Sigma = \hat h^{-1}(0) = \{(\theta_1(r), r) : r > 0\} \cup \{(\theta_2(r), r) : r > 0\}$.

Denoting $X^1 = (\varepsilon F_1(\theta, r) + \mathcal{O}(\varepsilon^2), -1)$ and $X^2 = (\varepsilon G_1(\theta, r) + \mathcal{O}(\varepsilon^2), -1)$, for $|\varepsilon| \neq 0$ sufficiently small we have

$$X^1\hat{h}(\theta_i(r), r).X^2\hat{h}(\theta_i(r), r) = \frac{(-1+\sqrt{1+4r^2})^2}{2r^4},$$

for $i = 1, 2$. Then $\hat{\Sigma}$ has only crossing regions. So hypotesis (H2) of Theorem 2 holds for system (8). Computing the averaged function f_1 we obtain

$$\begin{aligned} f_1(r) &= \int_{\theta_1}^{\theta_2} F_1(\theta, r)\, d\theta + \int_{\theta_2}^{\theta_1+2\pi} G_1(\theta, r)\, d\theta \\ &= g_0k_0(r) + g_1k_1(r) + g_2k_2(r) + g_3k_3(r) \end{aligned}$$

where $g_0 = -d_0+\delta_0$, $g_1 = a_1-d_2-\alpha_1+\delta_2$, $g_2 = \alpha_1+\delta_2$, $g_3 = a_1+d_2-\alpha_1-\delta_2$. Since the functions k_0, k_1, k_2, k_3 are linearly independent and $k_1(r)$ has constant sign for $r > 0$, then from Lemma 3 the function $f_1(r)$ can have for convenient values of the coefficients g_0, g_1, g_2, g_3 three simple zeros. Consequently the derivative at those zeros is nonzero, and by Remark 1 the Brouwer degree of $f_1(r)$ at those zeros is nonzero. So the hypothesis (H3) holds at these zeros. Then by Theorem 2 the proof of theorem follows. □

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References

[1] A. Andronov, A Vitt and S. Khaikin, *Theory of Oscillations*, Pergamon Press, Oxford, 1966.

[2] J. C. Artés, J. Llibre, J. C. Medrado and M. A. Teixeira, *Piecewise linear differential systems with two real saddles*, Math. Comp. Sim. **95** (2013), 13–22.

[3] M. di Bernardo, C. J. Budd, A. R. Champneys and P. Kowalczyk, *Piecewise-Smooth Dynamical Systems: Theory and Applications*, Appl. Math. Sci., vol. 163, Springer-Verlag, London 2008.

[4] D. C. Braga and L. F. Mello, *Limit Cycles ina a family of discontinuous piecewise linear differential systems with two zones in the plane*, Nonlinear Dynamics **73** (2013), 1283–1288.

[5] C. Buzzi, C. Pessoa and J. Torregrosa, *Piecewise linear pertubations of a linear center*, Discrete Continuous Dyn. Syst. **33** (2013), 3915–3936.

[6] B. Coll, A. Gasull and R. Prohens, *Bifurcation of limit cycles from two families of centers*, Dynamics of Continuous, Discrete and Impulsive Systems, Series **A**: Mathematical Analysis **12** (2005), 275–287.

[7] R. D. Euzébio, J. Llibre, *On the number of limit cycles in discontinuous piecewise linear differential systems with two pieces separated by a straight line*, J. Math. Anal. Appl. **424** (2015), 475–486.

[8] A. F. Filippov, *Differential Equations with Discontinuous Righthand Sides*, Mathematics and its Applications (Soviet Series), Kluwer Academic Publishers, Dortrecht, 1988.

[9] E. Freire, E. Ponce, F. Rodrigo and F. Torres, *Bifurcation sets of continuous piecewise linear system with two zones*, Int. J. Bifurcation and Chaos **8** (1988), 2073–2097.

[10] E. Freire, E. Ponce and F. Torres, *Canonical Discontinuous Planar Piecewise Linear Systems*, SIAM J. Applied Dynamical Systems **11** (2012), 181–211.

[11] E. Freire, E. Ponce and F. Torres, *A general mechanism to generate three limit cycles in planar Filippov systems with two zones*, Nonlinear Dynamics **78** (2014), 251–263.
[12] M. Han and W. Zhang, *On Hopf bifurcation in non-smooth planar systems*, J. Differential Equations **248** (2010), 2399–2416.
[13] S. M. Huan and X. S. Yang, *On the number of limit cycles in general planar piecewise linear systems of node-node types*, J. Math. Anal. Appl. **411** (2013), 340–353.
[14] S. M. Huan and X. S. Yang, *Existence of limit cycles in general planar piecewise linear systems of saddle-saddle dynamics*, Nonlinear Anal. **92** (2013), 82–95.
[15] L. Li, *Three crossing limit cycles in planar piecewise linear systems with saddle-focus type*, Electron. J. Qual. Theory Differ. Equ. 2014, No. 70, 14 pp.
[16] J. Llibre, A.C. Mereu and D.D. Novaes, *Averaging theory for discontinuous piecewise differential systems*, J. Differential Equations **258** (2015), 4007–4032.
[17] J. Llibre, D.D. Novaes and M.A. Teixeira, *Limit cycles bifurcating from the periodic orbits of a discontinuous piecewise linear differential center with two zones*, Int. J. Bifurcation and Chaos **25** (2015), 1550144, 11 pp.
[18] J. Llibre, M. Ordóñez and E. Ponce, *On the existence and uniqueness of limit cycles in a planar piecewise linear systems without symmetry*, Nonlinear Analysis Series B: Real World Applications **14** (2013), 2002–2012.
[19] J. Llibre and E. Ponce, *Three nested limit cycles in discontinuous piecewise linear differential systems with two zones*, Dynamics of Continuous, Discrete and Impulsive Systems, Serie B **19** (2012), 325–335.
[20] J. Llibre, M. A. Teixeira and J. Torregrosa, *Lower bounds for the maximum number of limit cycles of discontinuous piecewise linear differential systems with a straight line of separation*, Int. J. Bifurcation and Chaos **23** (2013), 1350066, 10 pp.
[21] N. G. Lloyd, *Degree Theory*, Cambridge University Press, Cambridge, 1978.
[22] R. Lum, L. O. Chua, *Global properties of continuous piecewise-linear vector fields. Part I: Simplest case in R^2*, Memorandum UCB/ERL M90/22, University of California at Berkeley, 1990.
[23] S. Shui, X. Zhang and J. Li, *The qualitative analysis of a class of planar Filippov systems*, Nonlinear Anal. *73* (2010), 1277–1288.

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