THE FREE EULER EQUATIONS REVISITED

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Abstract. We review from a different perspective the approach and solution to the torque-free Euler equations, also called the free asymmetric top equations. This article is intended to broaden the study of the asymmetric rigid body at the undergraduate level. This is an old but important integrable problem that has two first integrals: the energy and the angular momentum. We make two reductions to this problem. The first comes from introducing dimensionless variables and parameters in the equations of motion, reducing from three static parameters, the moments of inertia, to one called the asymmetry parameter κ, and from two original dynamic parameters of energy and angular momentum, to one called the energy parameter, e_0. Then we eliminate the time as the independent variable in the three autonomous Euler equations to obtain the equations of the trajectory in a space of dimension two, with non-autonomous differential equations. We focus on the geometric aspects of the trajectory of the Euler top: the parameter space is divided in six disjoint regions, whose boundaries are the degenerated cases. We give the solutions in these six regions and their boundaries depending on the values of κ and e_0, in terms of trigonometric functions of the cylindric angle ψ. Finally, we use these results to write the relation between the Euler angles of spin rotation ψ and nutation θ.

1. Historical Introduction

The study of the rigid body dates back to the XVIII century, with the pioneering works of Euler (1707-1783), who dedicated a large part of his life to study this problem, trying to explain and predict the motion of ships and their building (a historical review of Euler's works on the rigid body are in the article by Marquina et. al. [18]). Around 1736, Euler published in two large volumes his treatise *Mechanica sive motus scientia analytice exposita* (Mechanics or the science of motion, expounded analytically), where he proposes that the motion of a rigid body can be studied as two types of combined movements: one of translation around its center of gravity, and another of rotation of the body's orientation around an axis that passes through its center of gravity. The most simple rotation problem is when there are no torques, the so called *Euler free rigid body* or simply the *Euler top*. In 1755, Segner showed that every rigid body has three main axes where the inertial tensor is diagonal. Then, in the series of works that Euler wrote between 1758 and 1765 which culminate in the voluminous compendium *Theoria motus corporum rigorumum seu solidorum*, Euler used the principal axes system, and found the so called *Euler equations* and the *Euler angles*. He obtained the kinematic relations for the motion of a heavy rigid body and addressed the integrable problem of the free rigid body, reducing it to quadratures. The angular momentum in the inertial frame is a constant of motion, but in the body-fix frame it satisfies the Euler equations.

A geometric construction of the solution to the Euler equations for the free rigid body was given by Luis Poinsot [23] in 1851 using the *polhod* and *herpolhod*: the inertia ellipsoid

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