4-DIMENSIONAL ZERO HOPF BIFURCATION OF QUADRATIC POLYNOMIAL DIFFERENTIAL SYSTEMS, VIA AVERAGING THEORY

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Abstract. It is known that the maximum number of limit cycles that can bifurcate from a zero-Hopf equilibrium point of a quadratic polynomial differential system in dimension two is 3, and that in dimension three is at least 3. Here we prove that in dimension 4 at least 9 limit cycles can bifurcate in a zero-Hopf bifurcation of a quadratic polynomial differential system.

1. Introduction and statement of the main result

A Hopf bifurcation takes place at a singular point of a differential system when this changes its stability. More precisely, it is a local bifurcation which can appears when a singular point of a differential system having a pair of complex conjugate eigenvalues crosses the imaginary axis of the complex plane when we move the parameters of the differential system. At this crossing under convenient assumptions on the differential system, one or several small-amplitude limit cycles bifurcate from the singular point.

When the pair of complex eigenvalues are on the imaginary axis, i.e. they are of the form $\pm bi$, if the other eigenvalues are non-zero, we talk about a *Hopf bifurcation*, but if some of the other eigenvalues are zero, we say that we have a *zero-Hopf bifurcation*. Here we are interested in the study of the zero-Hopf bifurcations when all the eigenvalues different from the $\pm bi$ are zero, we denote such kind of zero-Hopf bifurcation a *complete zero-Hopf bifurcation*. While there is a well developed theory for studying the Hopf bifurcations (see for instance [4, 10]), such theory does not exist for the zero-Hopf bifurcations. For the zero-Hopf bifurcations there are only partial results.

The goal of this paper is to study how many small-amplitude limit cycles can bifurcate in a complete zero-Hopf bifurcation at a singular point of a quadratic polynomial differential system in function of the dimension of the system.

Bautin [1] in 1954 proved that at most 3 small-amplitude limit cycles can bifurcate in a Hopf bifurcation at a singular point of a quadratic polynomial differential system in $\mathbb{R}^2$. Note that in $\mathbb{R}^2$ the notions of Hopf bifurction, zero-Hopf bifurcation and complete zero-Hopf bifurcation coincide.

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Also using Bautin's result it is easy to show that at least 3 small-amplitude limit cycles can bifurcate in a zero-Hopf bifurcation at a singular point of a quadratic polynomial differential system in $\mathbb{R}^3$, for a proof of this last result using averaging theory see the paper [3]. Some other results related with the zero-Hopf bifurcation of quadratic polynomial differential system in $\mathbb{R}^3$ can be found for instance in [6, 8, 14]. Note that in $\mathbb{R}^3$ the notions of zero-Hopf bifurcation and complete zero-Hopf bifurcation coincide.

Here we shall prove that at least 9 limit cycles can bifurcate in a complete zero-Hopf bifurcation of a quadratic polynomial differential system in $\mathbb{R}^4$, this result is obtained using averaging theory of second order.

More precisely, we investigate the zero-Hopf bifurcation at a singular point, that without loss of generality we can assume at the origin of coordinates, of the following quadratic polynomial differential systems in $\mathbb{R}^4$

$$
\begin{array}{rcl}
\dot{x} & = & (a_1\varepsilon + a_2\varepsilon^2)x - (b + b_1\varepsilon + b_2\varepsilon^2)y + \displaystyle\sum_{j=0}^{2} \varepsilon^j X_j(x,y,z,w), \\
\dot{y} & = & (b + b_1\varepsilon + b_2\varepsilon^2)x + (a_1\varepsilon + a_2\varepsilon^2)y + \displaystyle\sum_{j=0}^{2} \varepsilon^j Y_j(x,y,z,w), \\
\dot{z} & = & (c_1\varepsilon + c_2\varepsilon^2)z + \displaystyle\sum_{j=0}^{2} \varepsilon^j Z_j(x,y,z,w), \\
\dot{w} & = & (d_1\varepsilon + d_2\varepsilon^2)w + \displaystyle\sum_{j=0}^{2} \varepsilon^j W_j(x,y,z,w),
\end{array} \tag{1}
$$

where

$$
\begin{array}{rl}
X_j(x,y,z,w) = & a_{j0}x^2 + a_{j1}xy + a_{j2}xz + a_{j3}xw + a_{j4}y^2 + a_{j5}yz + \\
& a_{j6}yw + a_{j7}z^2 + a_{j8}zw + a_{j9}w^2,
\end{array}
$$

$Y_j(x,y,z,w)$, $Z_j(x,y,z,w)$ and $W_j(x,y,z,w)$ have the same expression as $X_j(x,y,z,w)$ by replacing a_{ji} respectively by b_{ji}, c_{ji} and d_{ji} for $j = 0,1,2$ and $i = 0,1,\ldots,9$. The coefficients $a_{ij}, b_{ij}, c_{ij}, d_{ij}, a_1, a_2, b, b_1, b_2, c_1, c_2, c_3, d_1, d_2$ are real parameters with $b \neq 0$. Note that system (1) for $\varepsilon = 0$ at the origin has eigenvalues $\pm bi, 0, 0$. So for $\varepsilon = 0$ a *zero-Hopf bifurcation* can occur.

Our main result is:

Theorem 1. *The following statements hold.*

(a) *At most* 2 *limit cycles bifurcate from the origin of system* (1) *when* $\varepsilon = 0$ *by applying the averaging theory of first order, and this upper bound is reached.*
(b) *At most* 9 *limit cycles bifurcate from the origin of system* (1) *when* $\varepsilon = 0$ *by applying the averaging theory of second order, and this upper bound is reached.*

Statement (a) of Theorem 1 is proved in section 3. Statement (b) is proved in section 4. In section 2 we recall the averaging theory of first and second order as it was stated in [2]. This will be the main tool for proving Theorem 1.

2. The averaging theory of first and second order

The aim of this section is to present the averaging method of first and second order as it was developed in [2, 5, 7]. The following result is Theorem 4.2 of [2].

Theorem 2. *We consider the following differential system*

$$\dot{x}(t) = \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x) + \varepsilon^3 R(t,x,\varepsilon), \tag{2}$$

where $F_1, F_2 : \mathbb{R} \times D \to \mathbb{R}^n, R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ are continuous functions, T-periodic in the first variable, and D is an open subset of $\mathbb{R}^n$. Assume that the following hypotheses (i) and (ii) hold. We assume:

(i) *F_1, F_2, R are locally Lipschitz with respect to x, $F_1(t,.) \in C^1(D)$ for all $t \in \mathbb{R}$, and R is differentiable with respect to ε. We define f_1, f_2: $D \longrightarrow \mathbb{R}^n$ as*

$$\begin{aligned} f_1(z) = & \ \frac{1}{T}\int_0^T F_1(s,z)ds, \\ f_2(z) = & \ \frac{1}{T}\int_0^T \left[D_z F_1(s,z) \int_0^s F_1(t,z)dt + F_2(s,z) \right] ds. \end{aligned} \tag{3}$$

(ii) *For $V \subset D$ an open and bounded set and for each $\varepsilon \in (-\varepsilon_f, \varepsilon_f) \setminus \{0\}$, there exists $a \in V$ such that $f_1(a) + \varepsilon f_2(a) = 0$ and $d_B(f_1 + \varepsilon f_2, V, a) \neq 0$.*

Then for $|\varepsilon| > 0$ sufficiently small there exists a T-periodic solution $\varphi(\cdot, \varepsilon)$ of the system (2) such that $\varphi(0,\varepsilon) \to a$ when $\varepsilon \to 0$.

Where $d_B(f_1 + \varepsilon f_2, V, 0)$ denotes the Brouwer degree of the function $f_1 + \varepsilon f_2$ in the neighborhood V of zero. It is known that if the function $f_1 + \varepsilon f_2$ is C^1 then it is sufficient to check that $\det(D(f_1 + \varepsilon f_2(a_\varepsilon))) \neq 0$ in order to have that $d_B(f_1 + \varepsilon f_2, V, 0) \neq 0$, for more details see [9].

For additional information on the averaging theory see the books [11, 15].

3. Proof of statement (a) of Theorem 1

First we rescale the variables (x,y,z,w) doing the change of variables $(x,y,z,w) = (\varepsilon X, \varepsilon Y, \varepsilon Z, \varepsilon W)$, second we pass to cylindrical coordinates doing $(X,Y,Z,W) = (\varrho\cos\theta, \rho\sin\theta, \eta, \xi)$, and third we take the angle θ as the new independent variable. Then system (1) becomes into the normal form for applying the averaging theory.

Thus in the variables (ρ, ξ, η) system (1) writes

$$
\begin{aligned}
\frac{d\rho}{d\theta} &= \varepsilon F_{11}(\theta, \rho, \eta, \xi) + \varepsilon^2 F_{21}(\theta, \rho, \eta, \xi) + O(\varepsilon^3),\\
\frac{d\xi}{d\theta} &= \varepsilon F_{12}(\theta, \rho, \eta, \xi) + \varepsilon^2 F_{22}(\theta, \rho, \eta, \xi) + O(\varepsilon^3),\\
\frac{d\eta}{d\theta} &= \varepsilon F_{13}(\theta, \rho, \eta, \xi) + \varepsilon^2 F_{23}(\theta, \rho, \xi, \eta) + O(\varepsilon^3).
\end{aligned} \tag{4}
$$

Taking

$$
\begin{aligned}
x &= (\rho, \xi, \eta),\\
t &= \theta,\\
F_1(t, x) &= (F_{11}(\theta, \rho, \eta, \xi), F_{12}(\theta, \rho, \xi, \eta), F_{13}(\theta, \rho, \eta, \xi)),\\
F_2(t, x) &= (F_{21}(\theta, \rho, \eta, \xi), F_{22}(\theta, \rho, \xi, \eta), F_{23}(\theta, \rho, \xi, \eta)),
\end{aligned}
$$

and $T = 2\pi$, system (4) is equivalent to system (2). For $i = 1, 2, 3$ from (3) we have that $f_1 = (f_{11}, f_{12}, f_{13})$ where

$$
f_{1i}(\rho, \eta, \xi) = \frac{1}{2\pi} \int_0^{2\pi} F_{1i}(\theta, \rho, \eta, \xi) d\theta.
$$

Doing these computations we get that

$$
\begin{aligned}
f_{11}(\rho, \eta, \xi) &= \frac{1}{b}(\rho(2a_1 + (a_{03} + b_{06})\xi + (a_{02} + b_{05})\eta)) = 0,\\
f_{12}(\rho, \eta, \xi) &= \frac{1}{b}((c_{00} + c_{04})\rho^2 + 2(c_{09}\xi^2 + \eta(c_1 + c_{08}\xi + c_{07}\eta))) = 0,\\
f_{13}(\rho, \eta, \xi) &= \frac{1}{b}((d_{00} + d_{04})\rho^2 + 2(\xi(d_1 + d_{09}\xi) + d_{08}\xi\eta + d_{07}\eta^2)) = 0.
\end{aligned} \tag{5}
$$

By Theorem 2 the unique limit cycle which bifurcates from the origin of system (1) is provided from the unique real zero with $\rho > 0$ of system (5).

Solving the first equation of (5) we obtain the solution

$$
\xi = -\frac{2a_1 + (a_{02} + b_{05})\eta}{a_{03} + b_{06}}
$$

Then the second and the third equations become

$$\begin{aligned} g_{12} = \ & \frac{8a_1^2 c_{09}}{(a_{03}+b_{06})^2} + (c_{00}+c_{04})\rho^2 - \frac{2}{(a_{03}+b_{06})^2}(2a_{03}a_1c_{08}+ \\ & 2a_1b_{06}c_{08} - 4a_{02}a_1c_{09} - 4a_1b_{05}c_{09} - a_{03}^2c_1 - 2a_{03}b_{06}c_1 - b_{06}^2c_1)\eta + \\ & \frac{2}{(a_{03}+b_{06})^2}(a_{03}^2c_{07} + 2a_{03}b_{06}c_{07} + b_{06}^2c_{07} - a_{02}a_{03}c_{08} - a_{03}b_{05}c_{08} \\ & -a_{02}b_{06}c_{08} - b_{05}b_{06}c_{08} + a_{02}^2c_{09} + 2a_{02}b_{05}c_{09} + b_{05}^2c_{09})\eta^2, \end{aligned}$$

$$\begin{aligned} g_{13} = \ & \frac{4a_1(2a_1d_{09} - a_{03}d_1 - b_{06}d_1)}{(a_{03}+b_{06})^2} + (d_{00}+d_{04})\rho^2 - \frac{2}{(a_{03}+b_{06})^2} \\ & (2a_{03}a_1d_{08} + 2a_1b_{06}d_{08} - 4a_{02}a_1d_{09} - 4a_1b_{05}d_{09} + a_{02}a_{03}d_1 + \\ & a_{03}b_{05}d_1 + a_{02}b_{06}d_1 + b_{05}b_{06}d_1)\eta + \frac{2}{(a_{03}+b_{06})^2}(a_{03}^2d_{07} + \\ & 2a_{03}b_{06}d_{07} + b_{06}^2d_{07} - a_{02}a_{03}d_{08} - a_{03}b_{05}d_{08} - a_{02}b_{06}d_{08} \\ & -b_{05}b_{06}d_{08} + a_{02}^2d_{09} + 2a_{02}b_{05}d_{09} + b_{05}^2d_{09})\eta^2. \end{aligned}$$

Eliminating ρ^2 between these two equations we get a quadratic equation in η which has at most two solutions, but when we substitute one of these two solutions of η in $g_{12} = 0$ or $g_{13} = 0$, since there appears only ρ^2, one of the two possible solutions is negative. Hence these system at most has two solutions with $\rho > 0$, and there are examples with two. Therefore by Theorem 2 we deduce that system (1) has at most two limit cycles. This case has been studied in [8]. We give an example proving that the bound is reached

Now we shall provide an example of the result of statement (a) of Theorem 1 having 2 limit cycles bifurcating from a zero-Hopf bifurcation. Consider the quadratic polynomial differential system

$$\begin{aligned} \frac{dx}{dt} &= -\frac{3}{2}\varepsilon x - y - xz + xw, \\ \frac{dy}{dt} &= x - \frac{3}{2}\varepsilon y + 2yz + yw, \\ \frac{dz}{dt} &= 3\varepsilon z - x^2 + 3y^2 - 11z^2 + 4w^2 + 2zw, \\ \frac{dw}{dt} &= 3\varepsilon w + \frac{1}{2}x^2 + \frac{1}{2}y^2 - 6z^2 + w^2 + 2zw. \end{aligned} \tag{6}$$

The eigenvalues of the singular point $(0,0,0,0)$ of system (6) are $-\dfrac{3\varepsilon}{2} \pm i$ and 3ε of multiplicity 2. From system (5) we have for system (6) that

$$\begin{aligned} f_{11}(\rho,\eta,\xi) &= \frac{\rho}{2}[-3+\eta+2\xi] = 0, \\ f_{12}(\rho,\eta,\xi) &= \frac{1}{2}[\rho^2 + \eta(3-11\eta) + \xi(4\xi+2\eta)] = 0, \\ f_{13}(\rho,\eta,\xi) &= \frac{1}{2}[\rho^2 + 2\xi(3+\xi) + 2\eta(-6\eta+2\xi)] = 0. \end{aligned} \tag{7}$$

Solving system (7) there are only two solutions (ρ,η,ξ) with $\rho>0$, namely

$$P_{12}=\left(3\sqrt{\frac{6}{5}},\mp\frac{3}{\sqrt{5}},\frac{1}{2}(3\pm\frac{3}{\sqrt{5}})\right).$$

Since

$$\det\left(\frac{\partial(f_{11},f_{12},f_{13})}{\partial(\rho,\eta,\xi)}\right)\Bigg|_{(\rho,\xi,\eta)=P_{12}}=\pm\frac{162}{\sqrt{5}}\neq 0,$$

by Theorem 2 system (6) has two limit cycles for $\varepsilon\neq 0$ sufficiently small.

4. Proof of statement (b) of Theorem 1

The averaged function of first order $(f_{11}(\rho,\xi,\eta),f_{12}(\rho,\xi,\eta),f_{13}(\rho,\xi,\eta))$ is identically zero if and only if

$$\begin{array}{llll} a_1=0, & b_{06}=-a_{03}, & b_{05}=-a_{02}, & c_{09}=c_1=c_{08}=c_{07}=0,\\ d_{04}=-d_{00}, & d_1=d_{09}=d_{08}=d_{07}=0, & c_{04}=-c_{00}. & \end{array}$$

Under theses conditions we can apply the averaging theory of second order. Then from (3) we have $f_2=(f_{21},f_{22},f_{23})=(f_{21}(\rho,\eta,\xi),f_{22}(\rho,\eta,\xi),\ f_{23}(\rho,\eta,\xi)$ where

$$\begin{aligned}
f_{21}(\rho,\eta,\xi)=\;&\frac{\rho}{8b^2}(8a_2b+(a_{00}a_{01}+a_{01}a_{04}-2a_{00}b_{00}-b_{00}b_{01}+2a_{04}b_{04}-\\
&b_{01}b_{04}+a_{05}c_{00}+b_{02}c_{00}-a_{02}c_{01}+a_{06}d_{00}+b_{03}d_{00}-a_{03}d_{01})\rho^2+\\
&4b(a_{13}+b_{16})\xi+4(a_{01}a_{09}+2a_{09}b_{04}-2a_{00}b_{09}-b_{01}b_{09}+b_{08}c_{03}-\\
&a_{08}c_{06}+2b_{09}d_{03}-2a_{09}d_{06})\xi^2+4b(a_{12}+b_{15})\eta+4(a_{01}a_{08}+2a_{08}b_{04}-\\
&2a_{00}b_{08}-b_{01}b_{08}+b_{08}c_{02}+2b_{07}c_{03}-a_{08}c_{05}-2a_{07}c_{06}+\\
&2b_{09}d_{02}+b_{08}d_{03}-2a_{09}d_{05}-a_{08}d_{06})\xi\eta+4(a_{01}a_{07}+2a_{07}b_{04}-\\
&2a_{00}b_{07}-b_{01}b_{07}+2b_{07}c_{02}-2a_{07}c_{05}+b_{08}d_{02}-a_{08}d_{05})\eta^2),\\
f_{22}(\rho,\eta,\xi)=\;&\frac{1}{2b^2}(b(c_{10}+c_{14})\rho^2-(a_{06}c_{00}+b_{03}c_{00}-a_{03}c_{01}+b_{00}c_{03}+b_{04}c_{03}-\\
&c_{03}c_{05}-a_{00}c_{06}-a_{04}c_{06}+c_{02}c_{06}-c_{06}d_{03}+c_{03}d_{06})\rho^2\xi+2bc_{19}\xi^2-\\
&2(b_{09}c_{03}-a_{09}c_{06})\xi^3+2bc_2\eta-(a_{05}c_{00}+b_{02}c_{00}-a_{02}c_{01}+b_{00}c_{02}+\\
&b_{04}c_{02}-a_{00}c_{05}-a_{04}c_{05}+bc_{10}+bc_{14}-c_{06}d_{02}+c_{03}d_{05})\rho^2\eta\\
&+bc_{18}\xi\eta-2(b_{09}c_{02}+b_{08}c_{03}-a_{09}c_{05}-a_{08}c_{06})\xi^2\eta+2bc_{17}\eta^2-\\
&2(b_{08}c_{02}+b_{07}c_{03}-a_{08}c_{05}-a_{07}c_{06})\xi\eta^2-2(b_{07}c_{02}-a_{07}c_{05})\eta^3),\\
f_{23}(\rho,\eta,\xi)=\;&\frac{1}{2b^2}(b(d_{10}+d_{14})\rho^2+2bd_2\xi-(a_{06}d_{00}+b_{03}d_{00}-a_{03}d_{01}+\\
&c_{06}d_{02}+b_{00}d_{03}+b_{04}d_{03}-c_{03}d_{05}-a_{00}d_{06}-a_{04}d_{06})\rho^2\xi+2bd_{19}\xi^2-\\
&2(b_{09}d_{03}-a_{09}d_{06})\xi^3-(a_{05}d_{00}+b_{02}d_{00}-a_{02}d_{01}+b_{00}d_{02}+b_{04}d_{02}+\\
&c_{05}d_{02}-a_{00}d_{05}-a_{04}d_{05}-c_{02}d_{05}+d_{03}d_{05}-d_{02}d_{06})\rho^2\eta+2bd_{18}\xi\eta-\\
&2(b_{09}d_{02}+b_{08}d_{03}-a_{09}d_{05}-a_{08}d_{06})\xi^2\eta+2bd_{17}\eta^2-2(b_{08}d_{02}+b_{07}d_{03}-\\
&a_{08}d_{05}-a_{07}d_{06})\xi\eta^2-2(b_{07}d_{02}-a_{07}d_{05})\eta^3).
\end{aligned}$$

We isolate ρ^2 from the equation $f_{21}(\rho,\eta,\xi)=0$ and we substitute it in $f_{2i}(\rho,\eta,\xi)=0$ for $i=2,3$. Then we get two polynomials $(g_{22},g_{23})=(g_{22}(\eta,\xi),g_{23}(\eta,\xi))$ of the form

$$
\begin{aligned}
g_{22} &= C_0+C_1\eta+C_2\xi+C_3\eta^2+C_4\eta\xi+C_5\xi^2+C_6\eta^3+C_7\eta^2\xi+C_8\eta\xi^2+C_9\xi^3,\\
g_{23} &= D_0+D_1\eta+D_2\xi+D_3\eta^2+D_4\eta\xi+D_5\xi^2+D_6\eta^3+D_7\eta^2\xi+D_8\eta\xi^2+D_9\xi^3,
\end{aligned}
$$

where the coefficients C_j and D_j multiplied by the coefficient of ρ^2 in $f_{21}(\rho,\eta,\xi)$ are polynomials in the coefficients of system (1) except that they are divided by some power of b. We do not provide the explicit expressions of the coefficients C_i and D_i because some of them are huge and they need several pages for writing one of such huge coefficients.

Looking only at the coefficients of system (1) which appear in C_j and D_j we see that C_0, C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8, C_9, D_0, D_1, D_2, D_3, D_4, D_5, D_6, D_7, D_8, D_9 are all independent because pairwise contain different coefficients of system (1), with the exceptions of the coefficients C_7 and C_8, and D_7 and D_8 that share the same coefficients of system (1). But now looking directly at the explicit expressions of C_7 and C_8, and of D_7 and D_8 we observe that they are also independent.

In short, since all coefficients of the system $g_{22}(\eta,\xi)=0$ and $g_{23}(\eta,\xi)=0$ are independent they can be chosen in such a way that the number of real solutions of that system corresponds to the maximum number provided by the Bezout Theorem, i.e. nine solutions. See [12] for more details on the Bezout Theorem.

In summary, since we are interested in the solutions of system $f_{21}(\rho,\eta,\xi)=0$, $f_{22}(\rho,\eta,\xi)=0$, $f_{23}(\rho,\eta,\xi)=0$, having $\rho>0$, by Theorem 2 the averaging theory of second order can produce 9 limit cycles in a zero-Hopf bifurcation at the origin of system (1). This completes the proof of statement (b) of Theorem 1.

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