# ON THE NUMBER OF LIMIT CYCLES IN PIECEWISE PLANAR QUADRATIC DIFFERENTIAL SYSTEMS

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Abstract. We consider piecewise quadratic perturbations of centers of piecewise quadratic systems in two zones determined by a straight line through the origin. By means of expansions of the displacement map, we are able to find isolated zeros of it, without dealing with the unsurprising difficult integrals inherent in the usual averaging approach. We apply our technique to non-smooth perturbations of the four families of isochronous centers of the Loud family, $S_1$, $S_2$, $S_3$, and $S_4$, as well as to non-smooth perturbations of non-smooth centers given by putting different $S_i$'s in each zone. To show the coverage of our approach, we apply its first order, which recovers the averaging method of the first order, in perturbations of the already mentioned centers considering all the straight lines through the origin. Then we apply its second order to perturbations of the above centers for a specific oblique straight line. Here in order to argue we introduce certain blow-ups in the perturbative parameters. As a consequence of our study, we obtain examples of piecewise quadratic systems with at least 12 limit cycles. By analyzing two previous works of the literature claiming much more limit cycles we found some mistakes in the calculations. Therefore, the best lower bound for the number of limit cycles of a piecewise quadratic system is up to now the 12 limit cycles found in the present paper.

## 1. Introduction

Consider the class of planar polynomial differential systems of degree $n$. The maximum number of *limit cycles* that a system in this class can have is called the *Hilbert number*, denoted by $H(n)$. The problem of finding $H(n)$ remounts to Hilbert and his 16th problem which is, up to our knowledge, open until these days. Actually, although it is well known that $H(1) = 0$ and lower bounds for $H(n)$, $n \geq 2$, have been found over the years, upper bounds for it are still unknown for all $n \geq 2$. The search for $H(2)$ has been the object of intense study during the last century. The best-known lower bound for $H(2)$ was given by Shi [28], by means of an example of a quadratic differential system having 4 limit cycles, so that $H(2) \geq 4$. For the cubics, Li, Liu, and Yang [20] showed that $H(3) \geq 13$. Denoting by $M(n)$ the maximum number of limit cycles bifurcating from a singular point of a polynomial system of degree $n$ as a degenerate Hopf bifurcation, it is clear that $M(n)$ is a lower bound for $H(n)$. Bautin [2] showed that $M(2) = 3$. Zoladek [31, 32] proved that $M(3) \geq 11$, see also a simpler proof by Christopher [6]. In [30], Yu and Tian gave an example with 12 limit cycles surrounding a singularity for cubic systems, so that $M(3) \geq 12$. This proof has some gaps but was corrected by Giné, Gouveia, and Torregrosa in [13].

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