FLEXIBILITY OF ENTROPIES FOR PIECEWISE EXPANDING UNIMODAL MAPS

LLUÍS ALSEDÀ, MICHAŁ MISIUREWICZ, AND RODRIGO A. PÉREZ

ABSTRACT. We investigate the *flexibility* of the entropy (topological and metric) for the class of piecewise expanding unimodal maps. We show that the only restrictions for the values of the topological and metric entropies in this class are that both are positive, the topological entropy is at most $\log 2$, and the metric entropy is not larger than the topological entropy.

In order to have a better control on the metric entropy, we work mainly with topologically mixing piecewise expanding skew tent maps, for which there are only 2 different slopes. For those maps, there is an additional restriction that the topological entropy is larger than $\frac{1}{2}\log 2$.

Moreover, we generalize and give a different interpretation of the Milnor-Thurston formula connecting the topological entropy and the kneading determinant for unimodal maps.

1. Introduction

Recently an important program in Dynamical Systems was initiated by Anatole Katok. It concerns *flexibility*, that is, the idea that for a given class of dynamical systems, dynamical invariants (for instance entropies) can take arbitrary values, subject only to natural restrictions. Various results in this direction were obtained for instance in papers [E, EK, BKRH].

Here we investigate the family of piecewise expanding unimodal maps. While they are not smooth, they are piecewise smooth (in fact, the maps that we consider are piecewise linear). For those maps, by [LaY], there exists an absolutely continuous invariant probability measure. By [LiY], this measure is unique. Therefore we can consider its metric entropy (which is also equal to its Lyapunov exponent), as well as the topological entropy of the map. Both entropies are positive, topological entropy is at most $\log 2$, and by the Variational Principle, the metric entropy is not larger than the topological entropy. We will show (Theorem B) that those are the only restrictions for the values of those entropies.

In order to have a better control on the metric entropy, we will work mainly with piecewise expanding skew tent maps, for which there are only 2 different slopes. In

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particular, for the topologically mixing expanding skew tent maps we prove a version of a theorem on flexibility of entropies. For those maps, there is the additional restriction that the topological entropy is larger than $\frac{1}{2}\log 2$. Again, it turns out that there are no additional restrictions (Theorem A).

There are two basic things that we have to prove in order to get Theorem A. One is continuity of the density of the absolutely continuous invariant probability measure as a function of a map, and the other one is existence of maps with small metric entropy. For this, we need strong estimates on the density of this measure. Classical methods, initiated in [LaY], using the variation and integral, are difficult and give too weak estimates. In particular, in [BK] continuity of the density as a function of the map is proved only at maps for which the turning point is not periodic.

The problem with this classical approach is that it is difficult to trace the trajectory of a density under the iterates of the Frobenius-Perron operator. This is due to the fact that most points have two preimages and the value of the image of the density at a given point depends on the values (and derivatives, that are usually different) at those preimages. However, for skew tent maps there is an alternative to this procedure. Instead of looking at the whole density, we look only at its jumps (discontinuities). Those jumps propagate along one trajectory of the map, and it is easy to keep track of them.

As the old saying goes, *nihil sub sole novum*,[1] and this method has been employed by Ito, Tanaka and Nakada [ITN] over 40 years ago. They obtained a simple formula for the densities of absolutely continuous invariant measures for skew tent maps. Their results are not as widely known as they deserve, probably due to the fact that the term "skew tent map" was not used at that time.[2]

Besides proving our main theorems about flexibility and theorems that lead to them, we make an interesting observation. Namely, the formula of Milnor and Thurston [MT], connecting for unimodal maps the kneading sequence to the topological entropy, can be reinterpreted easily as the fact that for a tent map the sum of reciprocals of derivatives of all iterates of the map at the critical value is zero. It turns out that this is also true for skew tent maps. We wonder whether this can be translated back into a language involving some entropy-like quantities.

The paper is organized as follows. In Section 2 we give the basic definitions. In Section 3 we prove that if the positive slope of a mixing expanding skew tent map is large then the density of the absolutely continuous invariant probability measure is close to 1. In Section 4 we show that an additional assumption in the preceding section is necessary. In Section 5 we prove continuous dependence of this density on the map, while in Section 6 we prove continuous dependence of the metric entropy on the map. We do it for a larger class of maps, namely, we do not assume mixing. In Section 7 we modify the standard square root construction (basically inverse of the renormalization process) in order to stay in the class of piecewise expanding maps. Then, in Section 8 we prove our main theorems, and in Section 9 we make the observation we mentioned.

[1] Eccles. 1:10 (Vulg.)

[2] A search in the MathSciNet suggests that it was used for the first time in [MV].

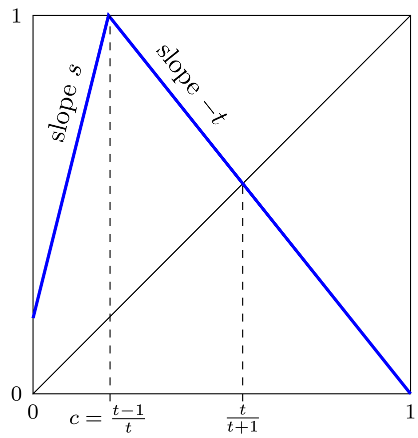

FIGURE 1. A skew tent map.

2. DEFINITIONS

An interval map $f : [0,1] \to [0,1]$ is called *piecewise expanding* if there is a finite partition of $[0,1]$ into smaller intervals, and on the closure of each of those smaller intervals f is of class C^2 and $|f'| \geq T$ for some constant $T > 1$. If f is unimodal, this partition can be finer than the partition into pieces of monotonicity (laps).

For a unimodal map $f : [0,1] \to [0,1]$ we assume that f is increasing on the left lap and decreasing on the right one. If c is the turning (critical) point for f then the *core* of f is the interval $[f^2(c), f(c)]$.

A *skew tent* map is a unimodal map which is linear (we will use this term in the sense of "affine"; this is a common terminology in real analysis) on each lap (see Figure 1). There are three popular models for skew tent maps. In the first one the map f is defined on some interval containing 0 in its interior, 0 is the turning point, and $f(0) = 1$ ([MV]). In the second and third ones f maps $[0,1]$ to itself. In the second model, $f(0) = f(1) = 0$ ([BK]). In the third model, $[0,1]$ is the core of f ([ITN]). We will use the third model. Thus, in particular, we will have $f(c) = 1$ and $f(1) = 0$. The slopes of f will be denoted by s (the left slope) and $-t$ (the right slope). The condition that f maps $[0,1]$ to itself translates to the condition $\frac{1}{s} + \frac{1}{t} \geq 1$.

The two main spaces of skew tent maps that we consider are the space $\mathfrak{Y}$ of all piecewise expanding skew tent maps and its subspace $\mathfrak{X}$ consisting of topologically mixing piecewise expanding skew tent maps. For computations it is good to remember that $s, t > 1$, that $c = (t-1)/t$, and that the fixed point is $t/(t+1)$ (there is a unique fixed point, except when $f(0) = 0$, and then we mean the other fixed point), see again Figure 1. In terms of slopes, each map of $\mathfrak{Y}$ is determined by slopes $s, t > 1$, subject to $\frac{1}{s} + \frac{1}{t} \geq 1$.

Lemma 2.1. *For a map $f \in \mathfrak{Y}$ the following properties are equivalent:*

(i) $f \in \mathfrak{X}$,
(ii) $f(0)$ *is to the left of the fixed point of* f,
(iii) $t > \frac{1}{s} + \frac{1}{t}$,
(iv) *topological entropy of* f *is larger than* $\frac{1}{2} \log 2$.

Proof. By Theorem 4.70 of [Ru], (i) implies (iv). To show that (iv) implies (ii), suppose that $f(0)$ is equal to or to the right of the fixed point. Then every point to the left of the fixed point is mapped to the right of the fixed point, and vice versa. The map f^2 restricted to the interval between 0 and the fixed point is unimodal, so its topological entropy is at most $\log 2$. Thus, the entropy of f is at most $\frac{1}{2}\log 2$.

To show that (ii) implies (i), note that $f(0) = 1 - \frac{s(t-1)}{t} = \frac{s+t-st}{t}$. Thus, by the formula (13) of [ITN] (in that paper our s, t are called a, b), our space $\mathfrak{X}$ is the same as the space D^* from [ITN] (formally, in that paper the authors speak about parameters, while we speak about maps). By Theorem 3.5 of [ITN], any map from that space is topologically exact, so it is topologically mixing. This proves that the properties (i), (ii) and (iv) are equivalent.

To show equivalence of (ii) and (iii), observe that $1 - \frac{s(t-1)}{t} < \frac{t}{t+1}$ is equivalent to $\frac{1}{t+1} < \frac{s(t-1)}{t}$, which is equivalent to $st^2 > t + s$, which is equivalent to (iii). □

We consider the spaces $\mathfrak{Y}$ and $\mathfrak{X}$ with the uniform (sup) topology. Parametrization via the absolute values of the slopes gives the same topology.

By [LaY], piecewise expanding maps have an absolutely continuous invariant probability measure (*acip* in short). By [LiY], if the map is unimodal, this measure is unique. We will use the notation μ for this measure and ϱ for its density. For μ multiplied by a positive constant (that is, an absolutely continuous invariant measure, that is finite, but not necessarily normed) we will use the acronym *acim*. Of course, by "absolutely continuous" we mean absolutely continuous with respect to the Lebesgue measure λ.

We will use notation from *kneading theory* (see, e.g., [CE] or [MT]). The *itinerary* of a point x is the sequence $A_0, A_1, \dots$, where $A_i = L$ if $f^i(x) < c$, $A_i = R$ if $f^i(x) > c$, and $A_i = C$ if $f^i(x) = c$. If there is no symbol C in this sequence, the itinerary is infinite. If there is a C, the sequence terminates at the first C. The central notion is the *kneading sequence* of the map f. There are two versions of it. The simpler one, which we will use in the proof of Lemma 3.3 and in Section 4, defines the kneading sequence as the itinerary of 1 (the image of the turning point). The other version, which we will use in all other places, defines the kneading sequence as the limit of the itineraries of x as x goes to 1 from the left. In this version, the symbol C does not appear in the kneading sequence.

3. Density is close to 1

The following theorem is proved in [ITN] (see Figure 2 for some examples of normalized densities obtained in this way).

Theorem 3.1. *For $f \in \mathfrak{Y}$, the function*

$$\widehat{\varrho} = \sum_{k=0}^{\infty} \frac{1}{(f^k)'(0)} \chi_{[f^k(0),1]} \tag{1}$$

is the density of an acim for f.

Remark 3.2. We have to explain how to understand the derivatives in (1) if 0 is periodic of period $p > 1$, because formally f^k is not differentiable at 0 for $k > p - 2$.

In [ITN], the authors use the chain rule and pretend that $f'(c) = -t$, so we can use the same convention in (1). However, it really does not matter. We have

$$\sum_{k=rp}^{rp+p-1} \frac{1}{(f^k)'(0)}\chi_{[f^k(0),1]} = \frac{1}{(f^{rp})'(0)}\sum_{k=0}^{p-1} \frac{1}{(f^k)'(0)}\chi_{[f^k(0),1]} = \frac{1}{(f^{rp})'(0)}\sum_{k=0}^{p-2} \frac{1}{(f^k)'(0)}\chi_{[f^k(0),1]}.$$

We could remove $\frac{1}{(f^{p-1})'(0)}\chi_{[f^{p-1}(0),1]}$ above, because $f^{p-1}(0) = 1$ and $[1,1]$ has Lebesgue measure zero. Therefore,

$$\sum_{k=0}^{\infty} \frac{1}{(f^k)'(0)}\chi_{[f^k(0),1]} = \left(\sum_{r=0}^{\infty} \frac{1}{(f^{rp})'(0)}\right)\sum_{k=0}^{p-2} \frac{1}{(f^k)'(0)}\chi_{[f^k(0),1]}.$$

Thus, $\sum_{k=0}^{p-2} \frac{1}{(f^k)'(0)}\chi_{[f^k(0),1]}$ is the density of an acim for f. This expression is well defined, because it does not involve $f'(c)$. Therefore,

$$\left(\sum_{r=0}^{\infty} \frac{1}{(f^{rp})'(0)}\right)\sum_{k=0}^{p-2} \frac{1}{(f^k)'(0)}\chi_{[f^k(0),1]} \tag{2}$$

is also an invariant density, *no matter how we define the values of* $(f^{rp})'(0)$. If we use the chain rule, we can define that always $f'(c) = -t$, or that always $f'(c) = s$, but also we can define that sometimes it is $-t$ and sometimes s.

Theoretically, there is a danger that the sum $\sum_{r=0}^{\infty} \frac{1}{(f^{rp})'(0)}$ is zero. However, the ratio of the rth and $(r+1)$st terms is equal to $(f^p)'(0)$ with $f'(c)$ defined as s or $-t$. We will see in the next lemma that this ratio is larger than 2. Consequently, our infinite sum is non-zero.

Lemma 3.3. *Assume that $f \in \mathfrak{Y}$ and 0 is periodic for f of period $p > 1$. Then*

$$\prod_{k=0}^{p-1} |f'(f^k(0))| \geq 1 + \min(s, t^2) > 2, \tag{3}$$

no matter whether $f'(c)$ is defined as s or $-t$.

Proof. Assume first that $f \in \mathfrak{X}$. In this case we prove that

$$\prod_{k=0}^{p-1} |f'(f^k(0))| \geq 1 + s > 2, \tag{4}$$

If the orbit of 0 includes a point from $(c, 1)$, then the left-hand side of (4) is at least st^2, which, by Lemma 2.1, is larger than $t + s$, which is larger than $1 + s$. The remaining possibility is that the kneading sequence of f is $RL^{p-2}C$. Then $f(0) \leq c$, so $s \geq \frac{1-c}{c} = \frac{1}{t-1}$. Therefore, $st \geq 1 + s$, and (4) follows.

Let us now consider the general situation of $f \in \mathfrak{Y}$. As we noticed in the proof of Lemma 2.1, if f is not mixing, then $f([0, c]) \subset [c, 1]$ and vice versa. Then f^2 restricted to $[0, c]$ maps $[0, c]$ to itself, and is a piecewise linear unimodal map. Thus, if we restrict it further, to its core I (see Section 2), we get a map which is almost a skew tent map, except that its domain is not $[0, 1]$ and that it has a minimum at the turning point. To remedy this, we rescale it by conjugating via the linear map from I onto $[0, 1]$ that reverses orientation. In such a way, we get a new skew tent map g, with slopes t^2 and $-st$. This process is called *renormalization* (see, e.g. [dMvS], Section II.5). If g is not mixing, we renormalize it again. This process has to stop, because the topological entropy of g is twice the topological entropy of f, and any map from $\mathfrak{Y}$ has positive entropy. Thus, by Lemma 2.1, maps from $\mathfrak{Y}$ which are renormalizable n times, but not $n+1$ times, have entropy larger than $\frac{1}{2^{n+1}}\log 2$ and smaller than or equal to $\frac{1}{2^n}\log 2$.

Now, if 0 is a periodic point of $f \in \mathfrak{Y}$ with period p and the n-th renormalization of f (call it g) is mixing, then by the chain rule, the product of the derivatives of f along the orbit of 0 is equal to the product of the derivatives of g along the orbit of 0, but the latter one is, as we already proved, at least 1 plus the left slope of g. If $n = 0$, then this slope is s, if $n = 1$ then it is t^2. If $n > 1$ then it is larger than s. This proves (3). □

By [K], $\widehat{\varrho}$ is bounded below by a positive constant on the support of the acim, which is the union of finitely many intervals. For $f \in \mathfrak{X}$, this support is just $[0,1]$.

Let us estimate the variation of $\widehat{\varrho}$ for $f \in \mathfrak{X}$. Since $f(0)$ is to the left of the fixed point of f, there exists $m \geq 0$ such that the kneading sequence of f is $RLR^mL\dots$. To see that, note that we allow $m = 0$. The first two symbols of the kneading sequence are always RL, so the only possibility that it does not start with RLR^mL is that it is RLR^∞, but then $f(0)$ is the fixed point. We will denote the space of all elements of $\mathfrak{X}$ with the kneading sequence $RLR^mL\dots$, where $m \leq n$, by $\mathfrak{X}_n$. Then $\mathfrak{X}$ is the union of the ascending sequence of subsets $\mathfrak{X}_n$.

Lemma 3.4. *If $f \in \mathfrak{X}_n$ and $s \geq t$ then*

$$\mathrm{Var}(\widehat{\varrho}) \leq \frac{n+1}{s} + \frac{2s+1}{s^2}. \tag{5}$$

Proof. Since variation of the sum is not larger than the sum of variations, by (1) (Theorem 3.1), we have

$$\mathrm{Var}(\widehat{\varrho}) \leq \sum_{k=1}^{\infty} \frac{1}{|(f^k)'(0)|}. \tag{6}$$

This series converges because the map f is piecewise expanding.

If $f \in \mathfrak{X}_n$, then we have $|(f^k)'(0)| \geq s$ for $k = 1, 2, \dots, n+1$ and $|(f^k)'(0)| \geq s^2 t^{k-2}$ for $k > n+1$, so

$$\mathrm{Var}(\widehat{\varrho}) \leq \frac{n+1}{s} + \frac{1}{s^2}\left(1 + \frac{1}{t} + \frac{1}{t^2} + \dots\right) = \frac{n+1}{s} + \frac{1}{s^2}\cdot\frac{t}{t-1}. \tag{7}$$

By Lemma 2.1, $f \in \mathfrak{X}$ is equivalent to $st^2 - s - t \geq 0$. Solving this inequality for t we get

$$t > \frac{1+\sqrt{1+4s^2}}{2s} \geq 1 + \frac{1}{2s}.$$

Since the function $t \mapsto \frac{t}{t-1}$ is decreasing, we get

$$\frac{t}{t-1} < \frac{1+\frac{1}{2s}}{\frac{1}{2s}} = 2s+1.$$

Therefore, for $f \in \mathfrak{X}_n$ we get the estimate (5). □

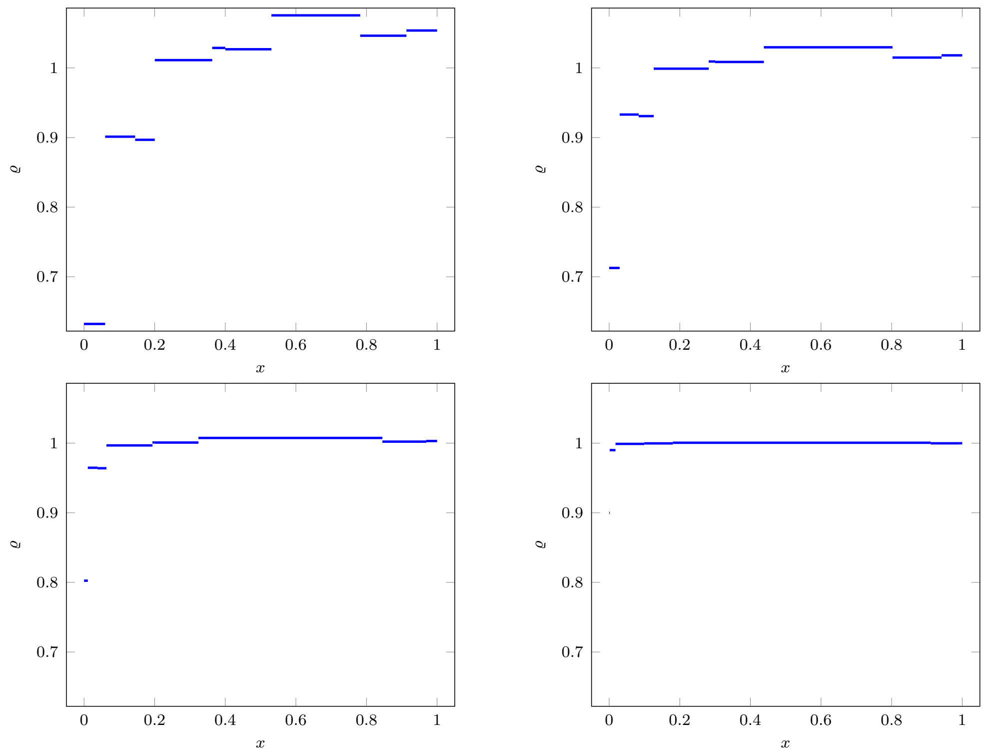

FIGURE 2. The graphs of the density ϱ for:
top left: $s = 2.35051\dots$ and $t = \frac{10}{6}$;
top right: $s = 3.2339\dots$ and $t = \frac{10}{7}$;
bottom left: $s = 4.9467\dots$ and $t = \frac{10}{8}$;
bottom right: $s = 9.9837\dots$ and $t = \frac{10}{9}$.

Now we normalize $\widehat{\varrho}$, that is, we set $\varrho = \widehat{\varrho}/\int \widehat{\varrho}$. Then ϱ is the density of the acip for f. Remember that by [LiY], the acip is unique.

Theorem 3.5. *For every n and $\varepsilon > 0$ there exists $s(\varepsilon, n)$ such that if $f \in \mathfrak{X}_n$ has slope $s \geq s(\varepsilon, n)$ then $1 - \varepsilon \leq \varrho \leq 1 + \varepsilon$ and* $\mathrm{Var}(\varrho) \leq \varepsilon$.

To illustrate Theorem 3.5 we show in Figure 2 four densities for maps from $\mathfrak{X}_0$ with larger and larger slope s.

Proof of Theorem 3.5. Let us start by estimating the value of $\widehat{\varrho}(0)$. If 0 is periodic of period p, then

$$|\widehat{\varrho}(0) - 1| \leq \sum_{i=1}^{\infty} \frac{1}{|(f^{ip})'(0)|} \leq \sum_{i=1}^{\infty} \frac{1}{s^i} = \frac{1}{s-1}. \tag{8}$$

If 0 is not periodic, then $\widehat{\varrho}(0) = 1$, so (8) also holds.

Now, the existence of the required $s(\varepsilon, n)$ follows from the fact that the limit as $s \to \infty$ is 0 on of the right hand side of both (5) (Lemma 3.4) and (8). $\square$

4. Necessity of spaces $\mathfrak{X}_n$

We want to justify the assumption that $f \in \mathfrak{X}_n$ in Theorem 3.5. For this we show that if we move through different classes $\mathfrak{X}_n$ then the limit density can be completely different (see Figure 3).

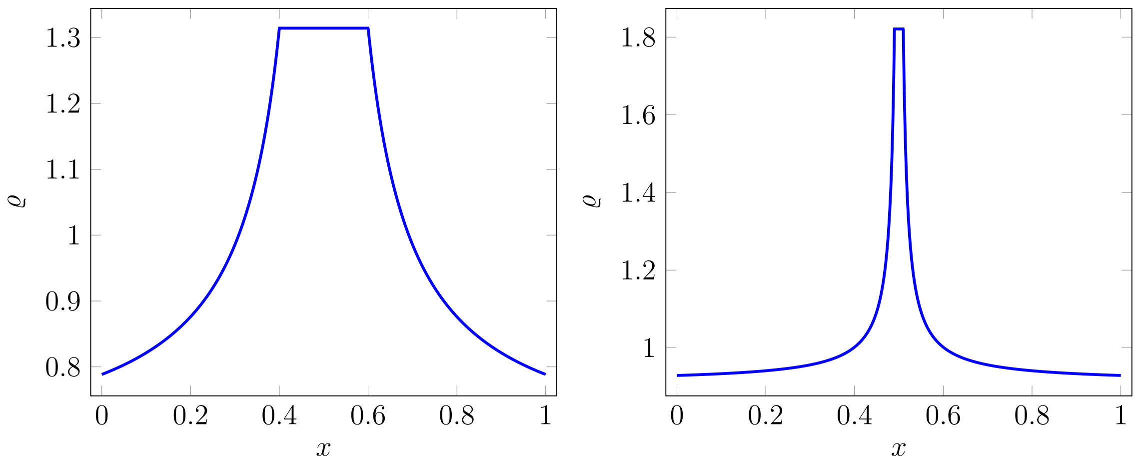

Figure 3. Two limit densities for $f(0) = a = 0.4$ (left picture) and $f(0) = a = 0.49$ (right picture).

Let us fix $a \in (0, 1/2)$ and take the skew tent map f with $f(0) = a$ and the kneading sequence $RLR^{2n}C$ (then the orbit of the turning point is the Štefan periodic orbit of period $2n + 3$). For $k = 0, 1, \ldots, 2n$, set

$$x_k = f^k(a) = \frac{t}{t+1} + \left(a - \frac{t}{t+1}\right)(-t)^k. \tag{9}$$

Let $\overline{\varrho}$ be the density of the acim, normalized by $\overline{\varrho}(0) = 1$ (so, since the only preimage of 0 is 1, and $\overline{\varrho}$ is invariant under the Frobenius-Perron operator, we have $\overline{\varrho}(1) = t$). The function $\overline{\varrho}$ is constant on $[a, f(a)]$ and has jumps

$$\frac{1}{s(-t)^k} = \frac{t-1}{(1-a)(-1)^k t^{k+1}}$$

at x_k, since

$$s = \frac{1-a}{c} = \frac{1-a}{1-\frac{1}{t}} = \frac{t(1-a)}{t-1}.$$

Let us concentrate on the right part of the interval (the situation on the left one is very similar). If $\overline{p}$ is the value of $\overline{\varrho}$ on $[a, f(a)]$, then the value of $\overline{\varrho}$ at x_{2k+1} (more precisely, the limit from the left) is

$$\begin{aligned}\overline{\varrho}(x_{2k+1}) &= \overline{p} - \sum_{j=0}^{k-1} \frac{t-1}{(1-a)t^{2j+2}} \\ &= \overline{p} - \frac{t-1}{(1-a)t^2} \cdot \frac{1-1/t^{2k}}{1-1/t^2} = \overline{p} - \frac{1}{(1-a)(t+1)}\left(1 - \frac{1}{t^{2k}}\right).\end{aligned}$$

By (9),

$$\frac{1}{t^{2k}} = \frac{t\left(\frac{t}{t+1} - a\right)}{x_{2k+1} - \frac{t}{t+1}},$$

so

$$\overline{\varrho}(x_{2k+1}) = \overline{p} - \frac{1}{(1-a)(t+1)} \left(1 - \frac{t\left(\frac{t}{t+1} - a\right)}{x_{2k+1} - \frac{t}{t+1}}\right).$$

Since $x_{2n+1} = 1$, we have

$$\overline{p} = t + \frac{1}{(1-a)(t+1)} \left(1 - \frac{t\left(\frac{t}{t+1} - a\right)}{1 - \frac{t}{t+1}}\right) = \frac{1}{1-a}.$$

Now let us go with n to infinity, keeping a constant. Then t goes to 1, $f(a)$ goes to $1-a$, and for every $\varepsilon > 0$ the points x_{2k+1} are ε-dense in $[1-a, 1]$ if n is large enough. Therefore, $\overline{\varrho}$ goes to the limit $\widetilde{\varrho}$, and for $x \geq 1-a$ we have

$$\widetilde{\varrho}(x) = \frac{1}{1-a} - \frac{1}{2(1-a)} \left(1 - \frac{\frac{1}{2} - a}{x - \frac{1}{2}}\right) = \frac{1}{2(1-a)} + \frac{1-2a}{4(1-a)} \cdot \frac{1}{x - \frac{1}{2}}.$$

Very similar computations give us

$$\widetilde{\varrho}(x) = \frac{1}{2(1-a)} + \frac{1-2a}{4(1-a)} \cdot \frac{1}{\frac{1}{2} - x}$$

for $x \leq a$.

In such a way we get

$$\widetilde{\varrho}(x) = \begin{cases} \frac{1}{2(1-a)} + \frac{1-2a}{4(1-a)} \cdot \frac{1}{\frac{1}{2}-x} & \text{if } x < a, \\ \frac{1}{1-a} & \text{if } a \leq x \leq 1-a, \\ \frac{1}{2(1-a)} + \frac{1-2a}{4(1-a)} \cdot \frac{1}{x-\frac{1}{2}} & \text{if } x > 1-a, \end{cases} \tag{10}$$

so in particular, $\widetilde{\varrho}$ is symmetric with respect to 1/2.

Now we have to normalize $\widetilde{\varrho}$ (that is, to divide it by its integral) in order to get the limit density ϱ. We have

$$\begin{aligned} \int \widetilde{\varrho}(x)\, dx &= 2 \int_0^a \left(\frac{1}{2(1-a)} + \frac{1-2a}{4(1-a)} \cdot \frac{1}{\frac{1}{2} - x}\right) dx + \frac{1-2a}{1-a} \\ &= \frac{a}{1-a} + \frac{1-2a}{1-a} + \frac{1-2a}{2(1-a)} \int_0^a \frac{1}{\frac{1}{2} - x}\, dx \\ &= 1 - \frac{1-2a}{2-2a} \log\left(\frac{1}{2} - x\right)\Big|_0^a = 1 - \frac{1-2a}{2-2a} \log(1-2a). \end{aligned}$$

Thus the limit density ϱ of the acip is given by the function from formula (10) divided by $1 - (1-2a)\log(1-2a)/(2-2a)$.

Observe that as a goes to 1/2 then the maximal value of ϱ (at the plateau) increases to 2. This motivates us to make the following conjecture, with weaker and stronger versions.

Conjecture 4.1. *There exists a constant K such that for every $f \in \mathfrak{X}$ the density of the acip for f is bounded above by K.*

Conjecture 4.2. *For every $f \in \mathfrak{X}$ the density of the acip for f is bounded above by 2.*

5. Continuous dependence of densities

The next thing to show is that the density ϱ of the acip depends continuously on the map. In this section we will stress the dependence on the map, so we will write ϱ_f, $\widehat{\varrho}_f$, etc.

For $\widehat{\varrho}_f = \widehat{\varrho}$ given by (1), let us look at its approximations $\widehat{\varrho}_{\ell,f}$, given by

$$\widehat{\varrho}_{\ell,f} = \sum_{k=0}^{\ell} \frac{1}{(f^k)'(0)} \chi_{[f^k(0),1]}. \tag{11}$$

We would like to prove that $\widehat{\varrho}_g$ as a function of g is continuous at any given f. Unfortunately, this is not true, because if 0 is periodic for f, the number $f'(c)$ is involved in the formula for $\widehat{\varrho}_f$, and it can change under arbitrarily small perturbations of f. We need to circumvent it using Remark 3.2. We will denote the ball of radius δ in $\mathfrak{Y}$ (in the sup metric), centered at f by $B(f,\delta)$. We will also denote the L^1 norm by $\|\cdot\|$.

Lemma 5.1. *Fix $f \in \mathfrak{Y}$ (with slopes s and $-t$) and $\varepsilon > 0$. Then there exists $\delta > 0$ such that for every $g \in B(f,\delta)$ there is a scalar $\gamma > \frac{\min(s,t^2)-1}{\min(s,t^2)+1}$ satisfying*

$$\|\widehat{\varrho}_g - \gamma\widehat{\varrho}_f\| < \varepsilon. \tag{12}$$

Proof. Suppose first that 0 is not periodic for f (or $f(0)=0$). There exist constants $T > 1$ and $\delta > 0$ such that if $g \in B(f,\delta)$ then the absolute values of the slopes of g are at least T. Then the sup distance between $\widehat{\varrho}_g$ and $\widehat{\varrho}_{\ell,g}$ is not larger than

$$\sum_{k=\ell+1}^{\infty} \frac{1}{T^k} = \frac{1}{T^\ell(T-1)}.$$

Let us choose ℓ so large that this is less than $\varepsilon/3$.

Observe that for $\alpha,\beta \in \mathbb{R}$ and $a,b \in [0,1]$ we have

$$\|\alpha\chi_{[a,1]} - \beta\chi_{[b,1]}\| \le |\alpha-\beta| + |a-b|\max(|\alpha|,|\beta|). \tag{13}$$

If δ is sufficiently small, then for each $k \le \ell$ the points $f^k(0)$ and $g^k(0)$ are as close to each other as we want, and the derivatives $(f^k)'(0)$ and $(g^k)'(0)$ are as close to each other as we want. Therefore, by (13), making δ sufficiently small yields

$$\sum_{k=1}^{\ell} |f^k(0) - g^k(0)| < \varepsilon/3,$$

so then $\|\widehat{\varrho}_{\ell,f} - \widehat{\varrho}_{\ell,g}\| < \varepsilon/3$. This proves that for a sufficiently small δ, if $g \in B(f,\delta)$, then $\gamma = 1$, and

$$\|\widehat{\varrho}_g - \widehat{\varrho}_f\| < \varepsilon. \tag{14}$$

Now we consider the case of 0 being periodic for f with period $p > 1$. In this case, it is not necessarily true that for a small δ, $(f^k)'(0)$ and $(g^k)'(0)$ are close to each other. However, this is true if we replace $\widehat{\varrho}_f$ and $\widehat{\varrho}_{\ell,f}$ by $\widehat{\widehat{\varrho}}_f$ and $\widehat{\widehat{\varrho}}_{\ell,f}$, where we change the values of $f'(f^{rp+p-2})$ to shadow the values of $g'(g^{rp+p-2})$ (look at the kneading sequence for g; if at the position $rp+p-2$ we see L, set $f'(f^{rp+p-2}(0)) = s$, otherwise set $f'(f^{rp+p-2})(0)) = -t$). Here we are using Remark 3.2. While always $f^{rp+p-2}(0) = c$, we are allowed to choose values for the derivative for different values of r. With this modification, instead of (14) we get (12) for some γ.

By Lemma 3.3, we have

$$1 - \sum_{r=1}^{\infty} \frac{1}{(1+\min(s,t^2))^r} \le \sum_{r=0}^{\infty} \frac{1}{(f^{rp})'(0)} \le 1 + \sum_{r=1}^{\infty} \frac{1}{(1+\min(s,t^2))^r},$$

independently of whether our choices for $f'(f^{rp+p-2}(0))$ were as for $\widehat{\varrho}_f$ or as for $\widehat{\widehat{\varrho}}_f$. Since the right and left expressions above equal $1-\frac{1}{\min(s,t^2)}$ and $1+\frac{1}{\min(s,t^2)}$ respectively, we obtain the uniform estimate $\gamma > \frac{\min(s,t^2)-1}{\min(s,t^2)+1}$. $\square$

Now we can prove the main result of this section.

Theorem 5.2. *The map $g \mapsto \varrho_g$ from $\mathfrak{Y}$ to $L^1([0,1])$ is continuous.*

Proof. We use Lemma 5.1, but we have to switch from $\widehat{\varrho}$ to $\varrho = \widehat{\varrho}/\int\widehat{\varrho} = \widehat{\varrho}/\|\widehat{\varrho}\|$. Given $f \in \mathfrak{Y}$ and $\varepsilon > 0$, let δ, g and γ be as in Lemma 5.1. We have

$$\|\varrho_f - \varrho_g\| = \left\| \frac{\widehat{\varrho}_f}{\|\widehat{\varrho}_f\|} - \frac{\widehat{\varrho}_g}{\|\widehat{\varrho}_g\|} \right\| \le \left\| \frac{\gamma\widehat{\varrho}_f}{\gamma\|\widehat{\varrho}_f\|} - \frac{\widehat{\varrho}_g}{\gamma\|\widehat{\varrho}_f\|} \right\| + \left\| \frac{\widehat{\varrho}_g}{\gamma\|\widehat{\varrho}_f\|} - \frac{\widehat{\varrho}_g}{\|\widehat{\varrho}_g\|} \right\|$$

$$= \frac{\|\gamma\widehat{\varrho}_f - \widehat{\varrho}_g\|}{\gamma\|\widehat{\varrho}_f\|} + \|\widehat{\varrho}_g\| \frac{\big|\|\widehat{\varrho}_g\| - \gamma\|\widehat{\varrho}_f\|\big|}{\gamma\|\widehat{\varrho}_f\| \cdot \|\widehat{\varrho}_g\|} = \frac{\|\gamma\widehat{\varrho}_f - \widehat{\varrho}_g\| + \big|\|\widehat{\varrho}_g\| - \gamma\|\widehat{\varrho}_f\|\big|}{\gamma\|\widehat{\varrho}_f\|} \le \frac{2\|\gamma\widehat{\varrho}_f - \widehat{\varrho}_g\|}{\gamma\|\widehat{\varrho}_f\|}.$$

Thus, by Lemma 5.1,

$$\|\varrho_f - \varrho_g\| \le \frac{2\varepsilon(\min(s,t^2)+1)}{(\min(s,t^2)-1)\|\widehat{\varrho}_f\|}.$$

This shows continuity of the map $g \mapsto \varrho_g$ at f. Since $f \in \mathfrak{Y}$ was arbitrary, this map is continuous on $\mathfrak{Y}$. $\square$

6. Continuous dependence of metric entropy

For $g \in \mathfrak{Y}$ denote by μ_g the acip for g.

Theorem 6.1. *The map $g \mapsto h_{\mu_g}(g)$ from $\mathfrak{Y}$ to $\mathbb{R}$ is continuous.*

Proof. By the Rohlin Lemma ([P, R]), we have

$$h_{\mu_f}(f) = \int \log|f'|\ d\mu_f = \int \log|f'|\,\varrho_f\, d\lambda,$$

where λ is the Lebesgue measure.

Fix $f \in \mathfrak{Y}$. The density ϱ_f is bounded above by some constant M. There are also constants N, η such that for every $g \in B(f,\eta)$ the function $\log|g'|$ is bounded above by N.

On the other hand, in view of Theorem 5.2 and since the map $g \mapsto \log|g'|$ from $\mathfrak{Y}$ to $L^1([0,1])$ is continuous, for every $\varepsilon > 0$ there exists $\delta \in (0,\eta)$ such that

$$\big\|\log|f'| - \log|g'|\,\big\| < \frac{\varepsilon}{2M}, \text{ and}$$
$$\|\varrho_f - \varrho_g\| < \frac{\varepsilon}{2N},$$

whenever $g \in B(f,\delta)$. Thus,

$$\begin{aligned}
\left|h_{\mu_f}(f) - h_{\mu_g}(g)\right| &= \left|\int \log|f'|\,\varrho_f\,d\lambda - \int \log|g'|\,\varrho_g\,d\lambda\right| \\
&\le \int \big|\log|f'| - \log|g'|\big|\,\varrho_f\,d\lambda + \int \log|g'|\,|\varrho_f - \varrho_g|\,d\lambda \\
&\le M\big\|\log|f'| - \log|g'|\big\| + N\|\varrho_f - \varrho_g\| < \varepsilon.
\end{aligned}$$

□

7. Rectangular root

To switch from a map g to a map with the topological and metric entropies halved, we need a square root procedure. This procedure is a kind of reverse to the renormalization procedure (see the proof of Lemma 3.3). The traditional square root procedure works as follows. Remember that when we say "linear" we mean what in algebra is called "affine".

Suppose we have a unimodal map g with an acip ν. Then we define a unimodal map G in the following way. On $[0,1/3]$, $G(x) = (2 + g(1-3x))/3$, on $[2/3,1]$, $G(x) = 1 - x$, on $[1/3,2/3]$ G is linear to make it continuous. Figure 4 shows g, G and G^2. For G^2 the intervals $[0,1/3]$ and $[2/3,1]$ are invariant, and on each of them G^2 is linearly conjugate to g. The map G maps each of them to the other one. All points of $[1/3,2/3]$, except the fixed point, are eventually mapped by iterates of G to $[0,1/3] \cup [2/3,1]$ (except in the case when all those points are fixed points of G^2). Therefore $h(G) = (1/2)h(g)$ and $h_\varkappa(G) = (1/2)h_\nu(g)$, where $\varkappa$ is the acip for G obtained from ν (we will define it rigorously in a moment).

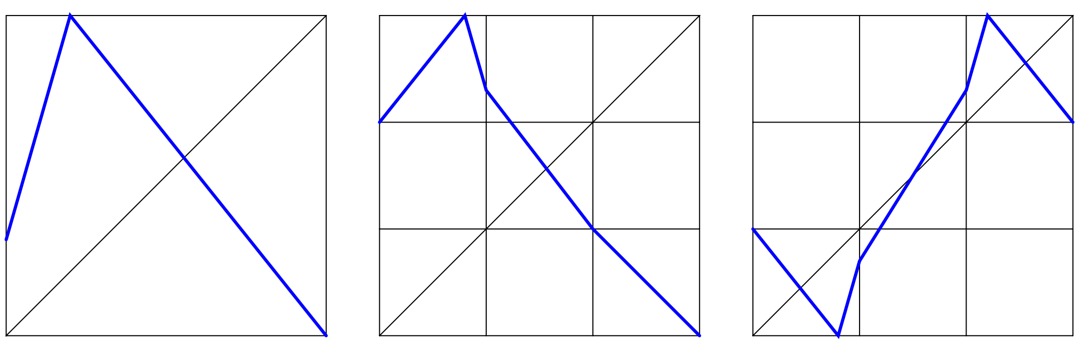

Figure 4. The square root procedure. Maps g, G and G^2.

The problem with this procedure is that even if we start with a piecewise expanding map, we end up with a map that has intervals on which the absolute value of the slope is 1. To modify this procedure in such a way that the resulting map is also piecewise expanding, let us look at the original procedure from the point of view of compositions of maps.

Let φ be the linear, orientation reversing map, that sends the interval $[0,1/3]$ onto the interval $[0,1]$. Then $\varphi(x) = 1 - 3x$ and $\varphi^{-1}(x) = (1-x)/3$. Similarly, let ψ be the linear orientation preserving map, sending the interval $[2/3,1]$ onto the interval

$[0,1]$. Then $\psi(x) = 3x - 2$ and $\psi^{-1}(x) = (2 + x)/3$. Now we see that on $[0, 1/3]$ we have $G = \psi^{-1} \circ g \circ \varphi$ and on $[2/3, 1]$ we have $G = \varphi^{-1} \circ \psi$. Since G sends $[0, 1/3]$ onto $[2/3, 1]$ and vice versa, on $[0, 1/3]$ we have $G^2 = \varphi^{-1} \circ g \circ \varphi$ and on $[2/3, 1]$ we have $G^2 = \psi^{-1} \circ g \circ \psi$. This explains why on both intervals G^2 is linearly conjugate to g.

This also allows us to give a precise definition of the measure $\varkappa$. Namely, $\varkappa$ is the average of the images of the measure ν under the maps φ^{-1} and ψ^{-1}. The equalities from the preceding paragraph show that since ν is invariant for g, $\varkappa$ is invariant for G.

If you look at Figure 4, you see nine small squares in a big square. If we change some of them into rectangles, our procedure will work better. Therefore we will call this procedure the *rectangular root*.

If $g(0) > 0$ then we define the rectangular root procedure by changing in the definition of φ the interval $[0, 1/3]$ to $[0, (1 + \varepsilon)/3]$ for some sufficiently small $\varepsilon > 0$ (see Figure 5). The slopes of G on $[0, (1 + \varepsilon)/3]$ are now equal to the slopes of g divided by $1 + \varepsilon$, so if ε is sufficiently small, their absolute values are still larger than 1. The slope of G on $[2/3, 1]$ is $-(1 + \varepsilon)$. The interval $[(1 + \varepsilon)/3, 2/3]$ is mapped by G onto a larger interval, so the absolute value of the slope is larger than 1 as well.

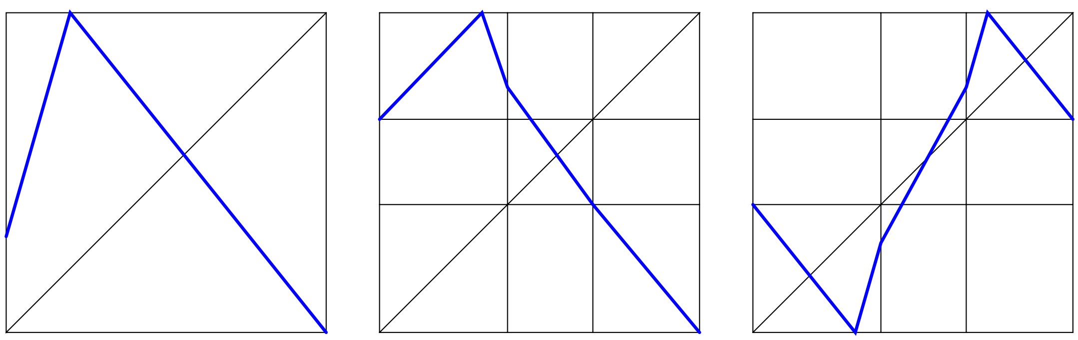

FIGURE 5. The rectangular root procedure when $g(0) > 0$. Maps g, G and G^2.

If $g(0) = 0$, then we simply remove the middle interval. That is, we take as φ the linear orientation reversing maps, sending the interval $[0, (1 + \varepsilon)/2]$ onto the interval $[0, 1]$, and as ψ the linear orientation preserving map, sending the interval $[(1+\varepsilon)/2, 1]$ onto the interval $[0, 1]$. Then we set $G = \psi^{-1} \circ g \circ \varphi$ on $[0, (1 + \varepsilon)/2]$ and $G = \varphi^{-1} \circ \psi$ on $[(1 + \varepsilon)/2, 1]$ (see Figure 6). As in the first case, if $\varepsilon > 0$ is sufficiently small, then G is piecewise expanding.

8. Main Theorems

The first flexibility result is about mixing skew tent maps (the space $\mathfrak{X}$).

Theorem A. *For every pair $a, b \in \mathbb{R}$ with $\frac{1}{2} \log 2 < a \leq \log 2$ and $0 < b \leq a$ there exists a piecewise expanding mixing skew tent map f for which $h(f) = a$ and $h_\mu(f) = b$, where μ is the acip for f.*

Proof. If $\log s_a = a$, then $f \in \mathfrak{X}$ with slopes s_a and $-s_a$ has topological entropy a. By [MV, Theorem C] and the subsequent remark, there exists a number $\gamma > 1$ and a

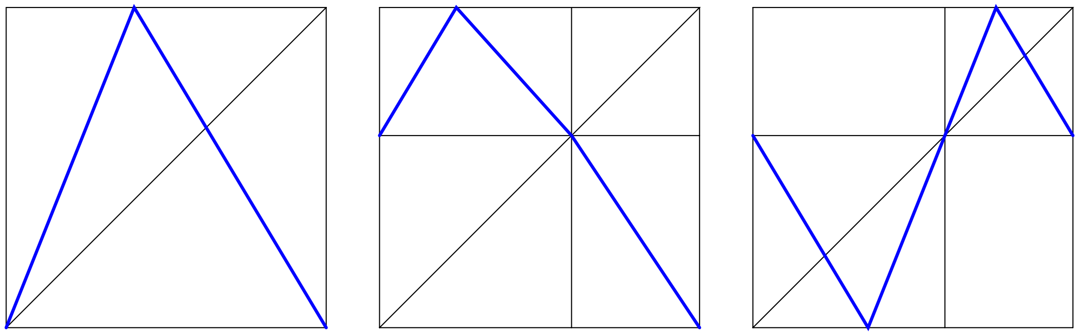

FIGURE 6. The rectangular root procedure when $g(0) = 0$. Maps g, G and G^2.

continuous decreasing function $\beta : (1, \gamma] \to [1, \infty)$ such that $\lim_{t \searrow 1} \beta(t) = \infty$ and for $g \in \mathfrak{X}$ we have $h(g) = a$ if and only if the slopes of g are $\beta(t), -t$ for $t \in (1, \gamma]$. This in particular implies that $s_a \le \gamma$ and $\beta(s_a) = s_a$. Moreover, by [MV, Theorem C], the skew tent maps with the slopes $\beta(t), -t$ have the same kneading sequence for all t, so we will be able to use Theorem 3.5 for them.

Let $f_t \in \mathfrak{X}$ be the function with slopes $\beta(t)$ and $-t$, and let μ_t be its acip. For $t = s_a$ this measure is also the measure with maximal entropy, so $h_{\mu_t}(f_t) = a$. As t goes to 1, then $\beta(t)$ goes to infinity and the turning point c_t goes to 0, so by Theorem 3.5 $\mu_t([0, c_t])$ goes to 0. Since the partition of $[0, 1]$ into $[0, c_t]$ and $(c_t, 1]$ is a generator, this implies that $h_{\mu_t}(f_t)$ goes to 0. Therefore, by Theorem 6.1 and continuity of the function β, there exists t such that $h_{\mu_t}(f_t) = b$. □

The next theorem shows flexibility of entropies for piecewise expanding unimodal maps.

Theorem B. *For every pair $a, b \in \mathbb{R}$ with $0 < a \le \log 2$ and $0 < b \le a$ there exists a piecewise expanding unimodal map f for which $h(f) = a$ and $h_\mu(f) = b$, where μ is the acip for f.*

Proof. If $a > \frac{1}{2} \log 2$, this follows from Theorem A. If $a \le \frac{1}{2} \log 2$, there is $n \ge 1$ such that $\frac{1}{2} \log 2 < 2^n a \le \log 2$. Then use Theorem A to find $g \in \mathfrak{X}$ with $h(g) = 2^n a$ and $h_\nu(g) = 2^n b$, where ν is the acip for g. Finally, use n times the rectangular root procedure (see Section 7) to get the desired f. Each time we use this procedure, both topological and metric entropy get divided by 2. Then f is a piecewise expanding unimodal map with $h(f) = a$ and $h_\mu(f) = b$. □

The following proposition shows that in Theorem B we cannot replace unimodal maps by skew tent maps.

Proposition 8.1. *There is no piecewise expanding skew tent map with topological entropy $\frac{1}{4} \log 2$ and metric entropy smaller than $\frac{1}{4} \log \frac{16}{15}$.*

Proof. Let $f \in \mathfrak{Y}$ have topological entropy $h(f) = \frac{1}{4} \log 2$. This means that f is twice renormalizable (see the proof of Lemma 3.3). If the slopes of f are s and $-t$, after two renormalizations we get a skew tent map g with slopes $s^2 t^2$ and $-st^3$ and entropy $\log 2$, linearly conjugate to f^4 restricted to an invariant interval.

Topological entropy $\log 2$ for a unimodal map is equivalent to this map having the kneading sequence RL^∞, so for a skew tent map means that the sum of reciprocals of the absolute values of the slopes is 1 It is well known (and vary easy to check) that then the Lebesgue measure is invariant, so the acip is the Lebesgue measure. Thus, we have $s + t = s^2t^3$. Remember that we assume $s, t > 1$.

Set $T = \max(s,t)$. We have $T^2 < s^2t^3 = s + t \leq 2T$, so $T < 2$. Hence, absolute values of both slopes of the second renormalization g are smaller than 16. Therefore $1/16 < c < 15/16$, so either $\lambda([0,c]) > 1/16$ or $\lambda((c,1]) > 1/16$. Then,

$$h_\lambda(g) > -\frac{1}{16}\log\frac{1}{16} - \frac{15}{16}\log\frac{15}{16} = \log 16 - \frac{15}{16}\log 15 > \log 16 - \log 15 = \log\frac{16}{15}.$$

Thus, if μ is the acip for f, then $h_\mu(f) = \frac{1}{4}h_\lambda(g) > \frac{1}{4}\log\frac{16}{15}$. □

9. An interesting observation

Hidden in [ITN] is the formula called by the authors "f-expansion". Namely, for $f \in \mathfrak{Y}$ and $x \in [0,1]$ we have, (translating to our notation)

$$x = 1 - \frac{1}{t}\sum_{k=0}^{\infty}\frac{1}{(f^k)'(x)}. \tag{15}$$

Taking $y > c$ for which $f(y) = x$, we can reinterpret it as

$$f(y) = \sum_{k=0}^{\infty}\frac{1}{(f^k)'(y)}. \tag{16}$$

However, since (16) does not work for $y < c$, it does not look important. On the other hand, if we take in (15) $y = 1$, we get

$$\sum_{k=0}^{\infty}\frac{1}{(f^k)'(1)} = 0, \tag{17}$$

which is much more interesting.

If $s = t$, then the left-hand side of (17) is the value of the Milnor-Thurston kneading determinant at $1/s$. However, s is the exponential of the topological entropy of f. Thus, formula (17) is a generalization and a different interpretation of the Milnor-Thurston formula connecting the topological entropy and the kneading determinant for unimodal maps.

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