Thurston Spider Maps for Chebyshev Polynomials Approximating Cosine

Schinella D'Souza
University of Michigan

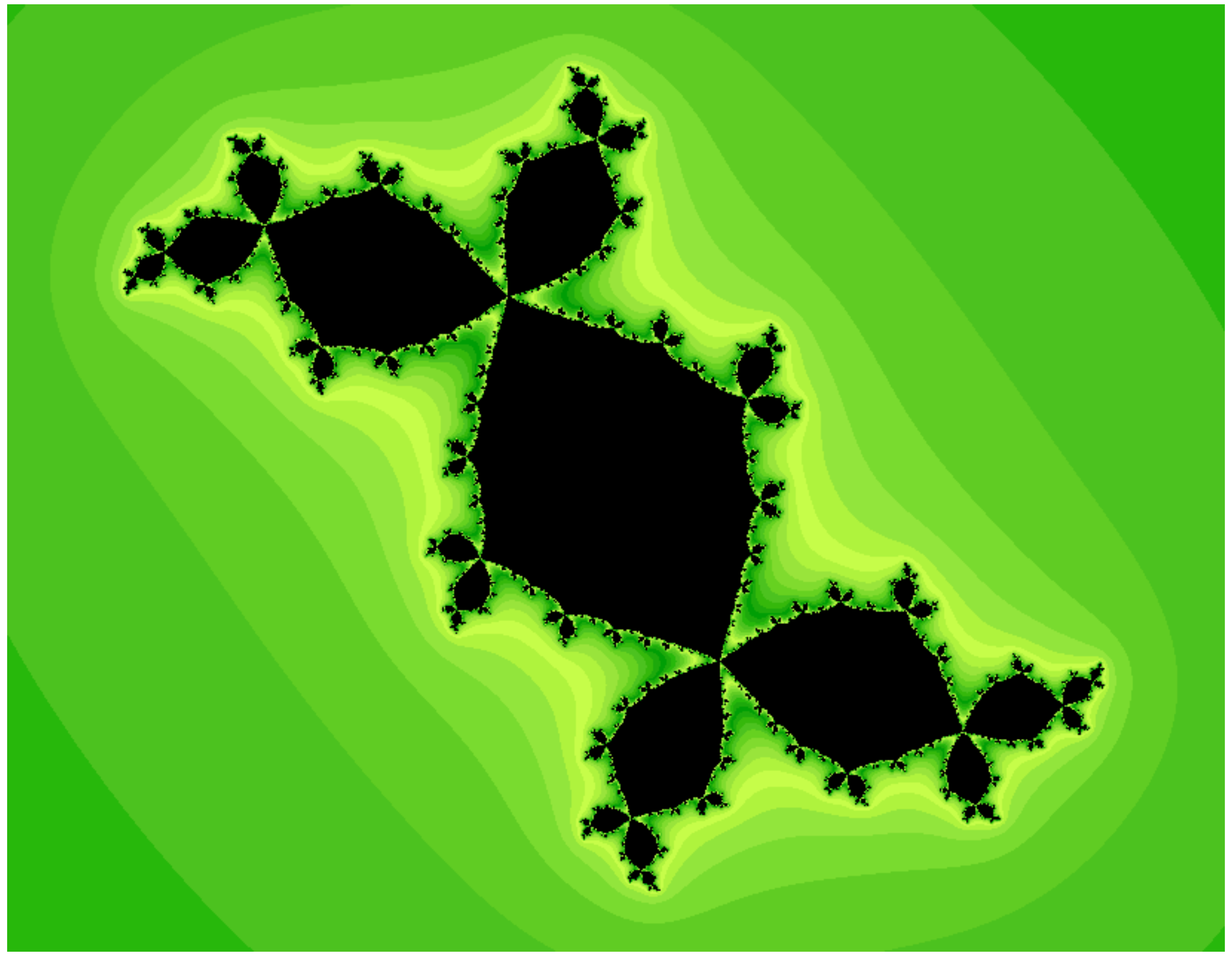
Summary

- Introduce a parametrized family of Chebyshev polynomials that approximate cosine
- Construction of Thurston spider map for each polynomial
- Conditions for when map is unobstructed

Background Definitions

Polynomial $p : \mathbb{C} \to \mathbb{C}$, deg$(p) \geq 2$:

- *Filled Julia Set:* $K_p := \{z \in \mathbb{C} \,|\, z, p(z), p^{\circ 2}(z), \ldots \text{ bounded}\}$
- *Julia Set:* $J(p) := \partial K_p$
- *Fatou Set:* $\mathbb{C} - J(p)$



Background Definitions

- *Thurston map:* branched map $f : S^2 \to S^2$ of degree ≥ 2 with $|P_f| < \infty$

  $P_f = \bigcup_{n \geq 1} f^{\circ n}(\mathrm{Crit}_f)$

- *Thurston equivalence (TE):*

$$\begin{array}{ccc} S^2 & \xrightarrow{\text{homeo } \varphi} & S^2 \\ f \downarrow & \circlearrowleft & \downarrow g \\ S^2 & \xrightarrow[\text{homeo } \psi]{} & S^2 \end{array}$$

$\varphi|_{P_f} = \psi|_{P_f}$

φ, ψ isotopic rel P_f

- *Thurston obstruction:* type of multicurve that is a barrier to TE to a rational function

Chebyshev Polynomials

Chebyshev polynomial: T_n such that $T_n(\cos\theta) = \cos n\theta$

$$T_1(z) = z$$

$$T_2(z) = 2z^2 - 1$$

$$T_3(z) = 4z^3 - 3z$$

$$T_4(z) = 8z^4 - 8z^2 + 1$$

$$T_5(z) = 16z^5 - 20z^3 + 5z$$

A Family of Chebyshev Polynomials

Chebyshev polynomial: T_n such that $T_n(\cos\theta) = \cos n\theta$

Family of Interest: $f_{2n,\lambda}(z) = \lambda(-1)^n T_{2n}\left(\frac{z}{2n}\right)$ where $n \in \mathbb{Z}_+, \quad \lambda \in \mathbb{C}\backslash\{0\}$

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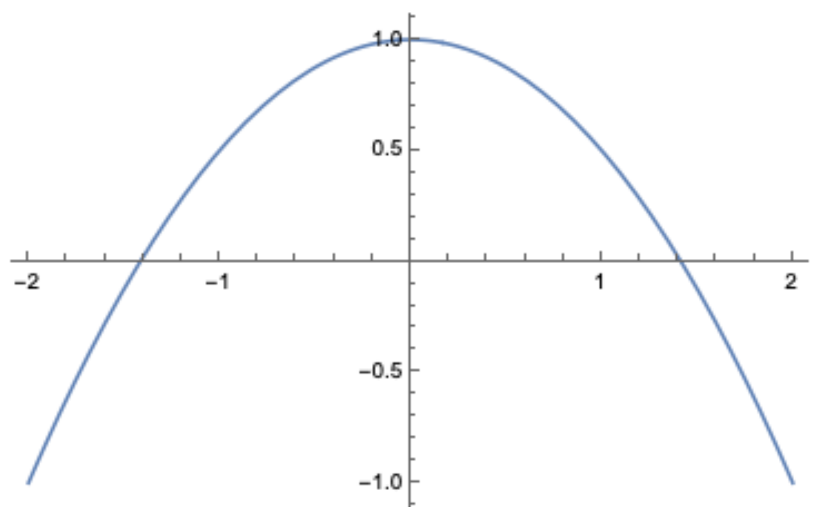
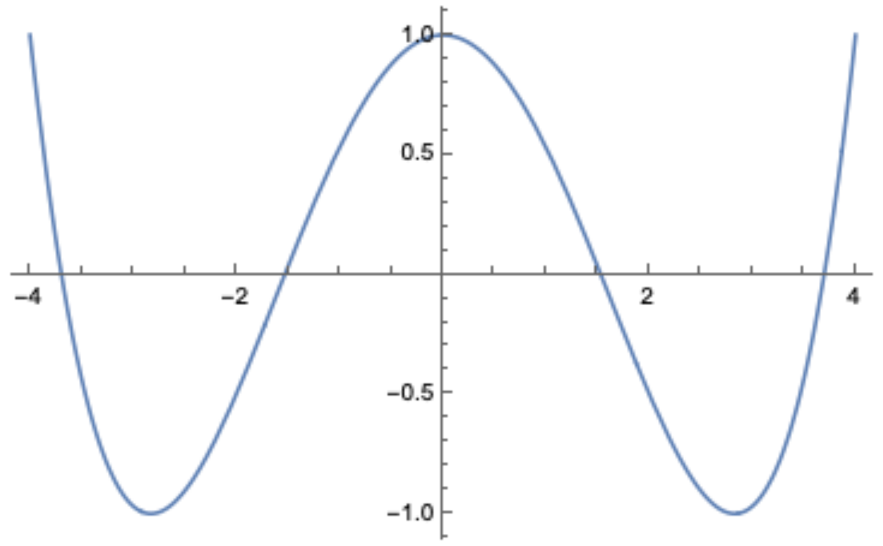
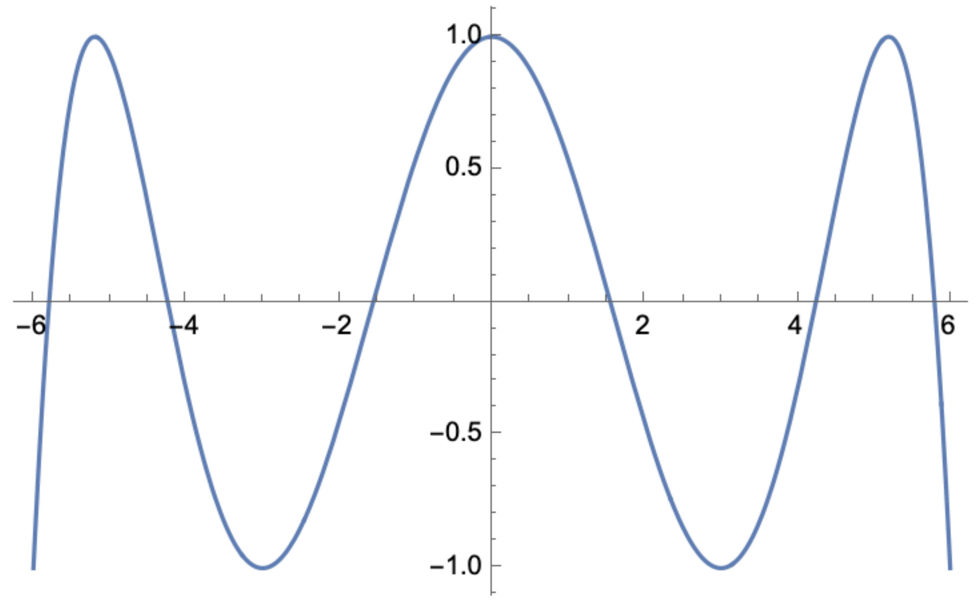


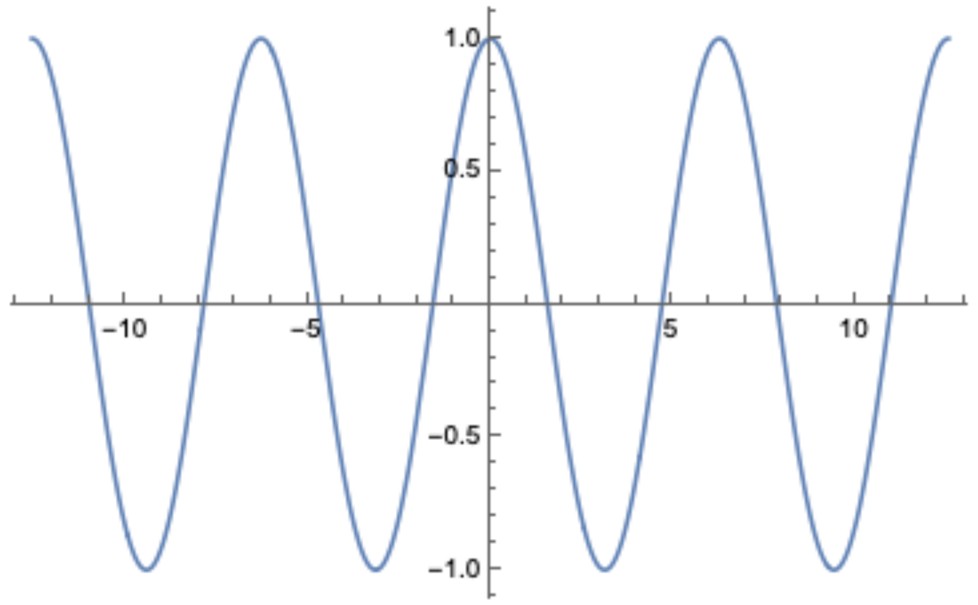
$n = 1, \quad \lambda = 1$



$n = 2, \quad \lambda = 1$



$n = 3, \quad \lambda = 1$

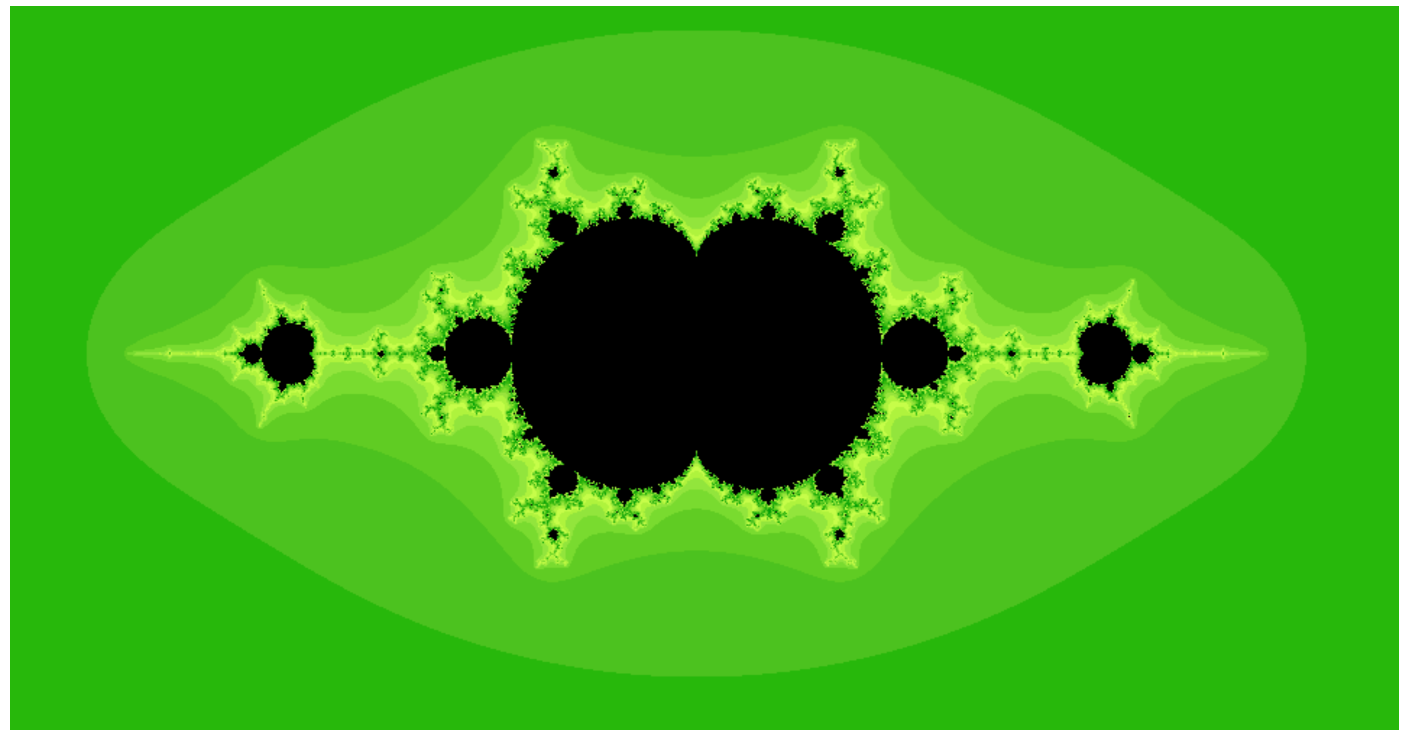


$z \mapsto \cos z$

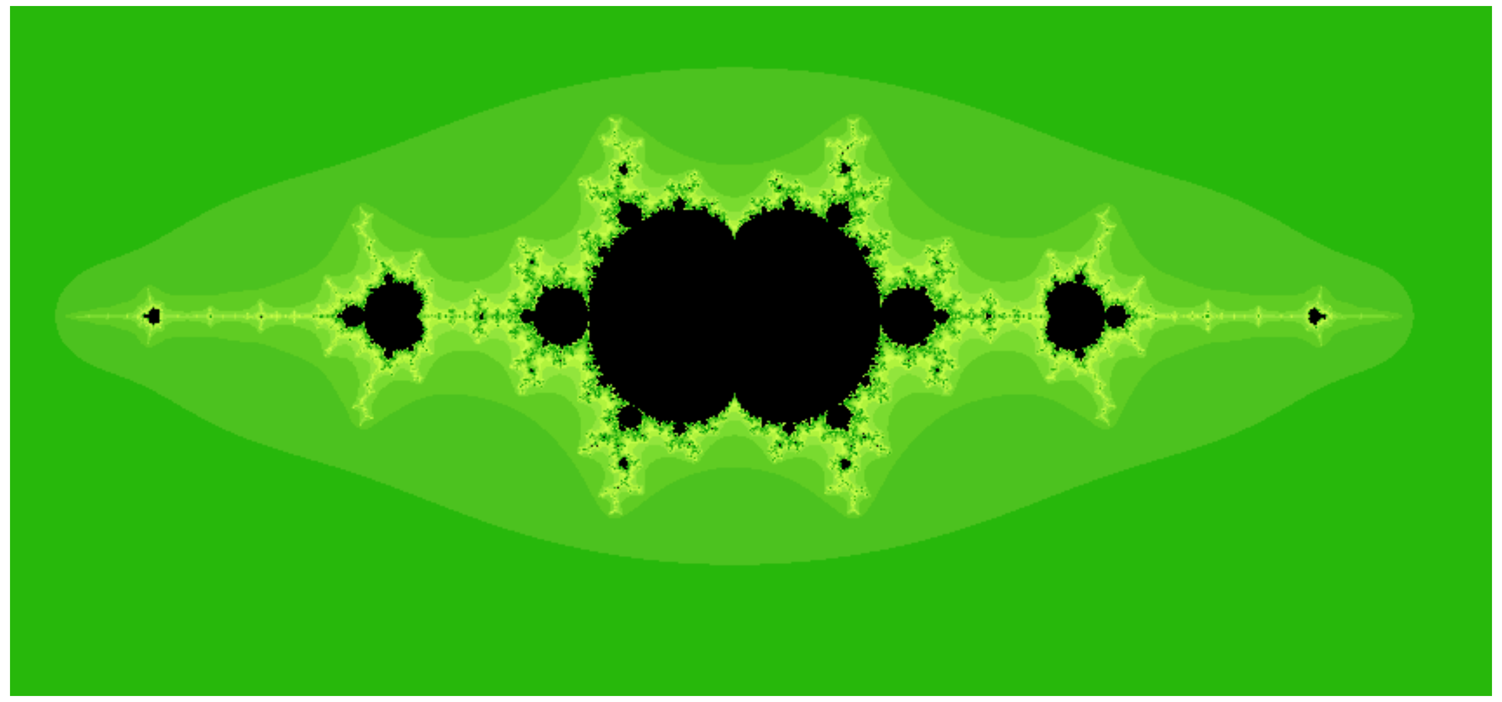
A Family of Chebyshev Polynomials

Chebyshev polynomial: T_n such that $T_n(\cos\theta) = \cos n\theta$

Family of Interest: $f_{2n,\lambda}(z) = \lambda(-1)^n T_{2n}\left(\frac{z}{2n}\right)$ where $n \in \mathbb{Z}_+, \quad \lambda \in \mathbb{C}$

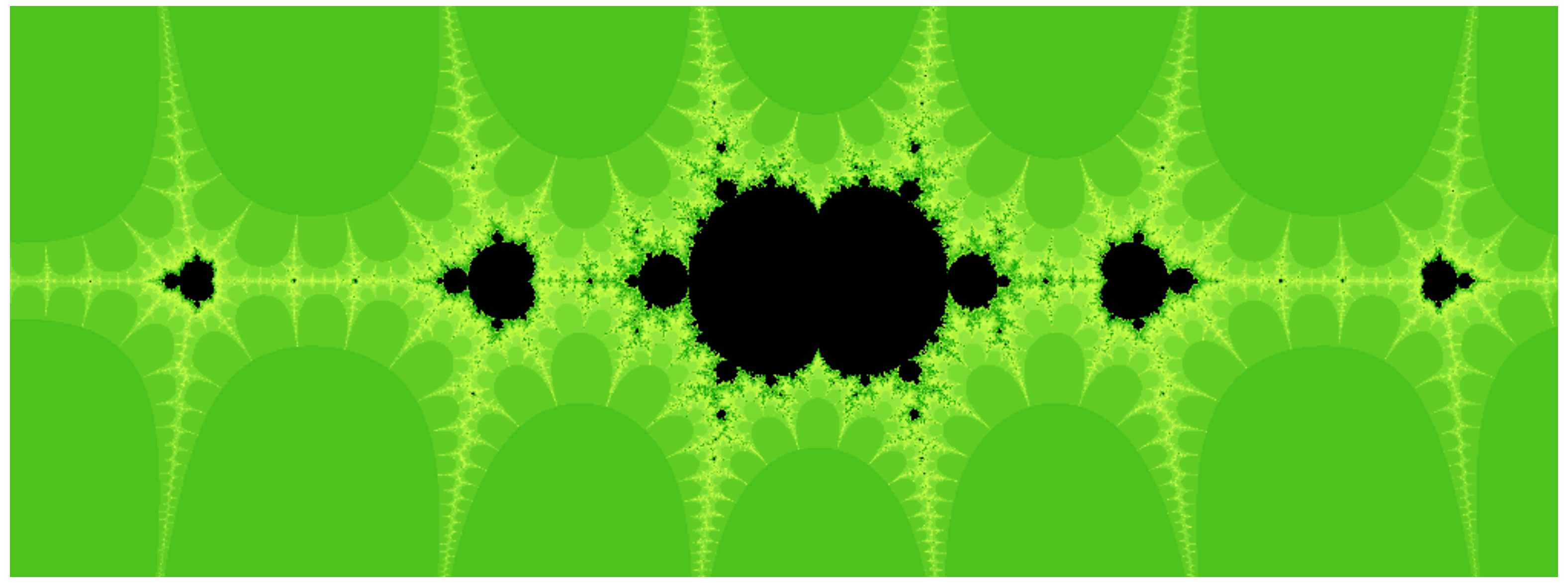


$n = 2$



$n = 3$

...



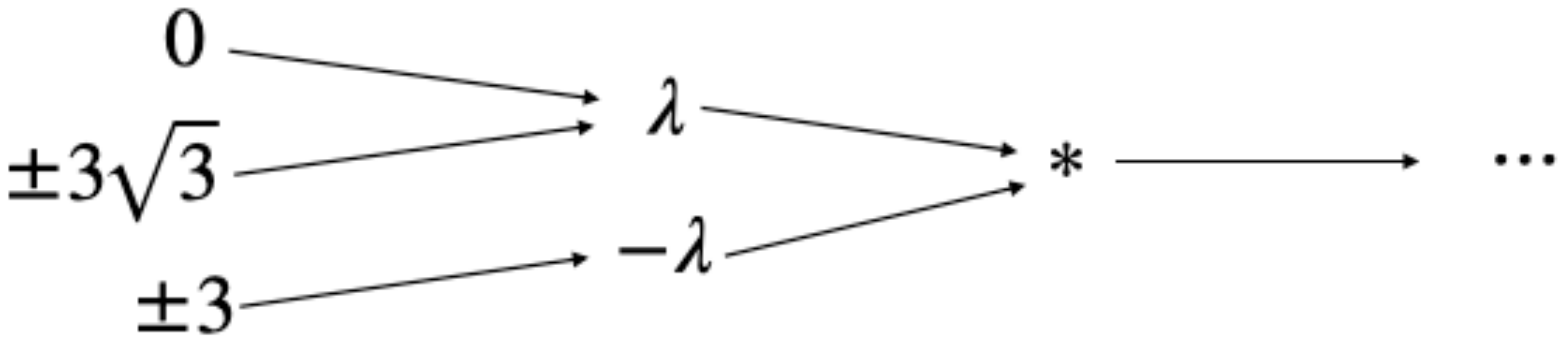
Properties of this Family

$$f_{2n,\lambda}(z) = \lambda(-1)^n T_{2n}\left(\frac{z}{2n}\right)$$

Critical Points: $2n-1$ of the form $z = 2n\cos\left(\frac{k\pi}{2n}\right) \in (-2n,\ 2n)$

Critical Values: $\pm\lambda$

Critical Orbits: For example, for $n = 3$



Some Results and References

- Bielefeld, Fisher, Hubbard (1992): Classifying critically preperiodic polynomials
- Hubbard, Schleicher (1994): Spider algorithm for quadratic polynomials
- Mukundan (2023): Dynamical approximation of PSF exponentials
- Mukundan, Prochorov, Reinke (2023): Dynamical approximation of PSF entire maps

Thurston Spider Map: Quadratics

Quadratic polynomial: $f_{2,\lambda}(z) = -\lambda T_2\left(\frac{z}{2}\right)$ $\quad 0 \mapsto \lambda \mapsto \cdots$

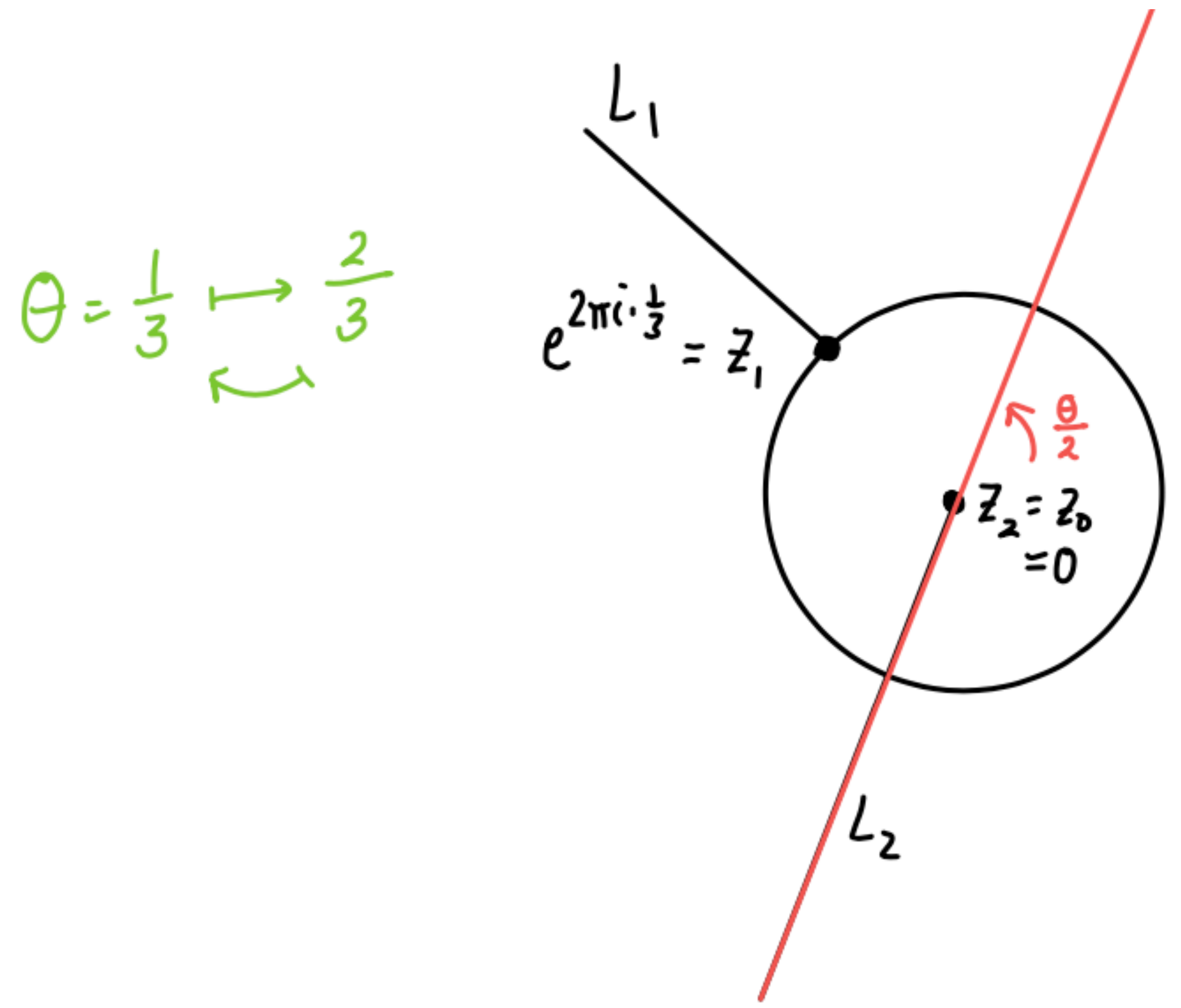
Let $\theta \in \mathbb{Q}/\mathbb{Z}$ periodic of period k.

i) *θ-Spider:*

$$\mathbb{S}_\theta := \{re^{2\pi i\cdot 2^j\theta} \mid r \geq 1, \quad j \in \mathbb{N}\} \cup \{re^{2\pi i 2^k\theta} \mid r \geq 0\} \cup \{\infty\}$$

ii) *Extended θ-Spider:*

$$\tilde{\mathbb{S}}_\theta := \mathbb{S}_\theta \cup \{te^{2\pi i\cdot\theta/2} \mid t \in \mathbb{R}\}$$

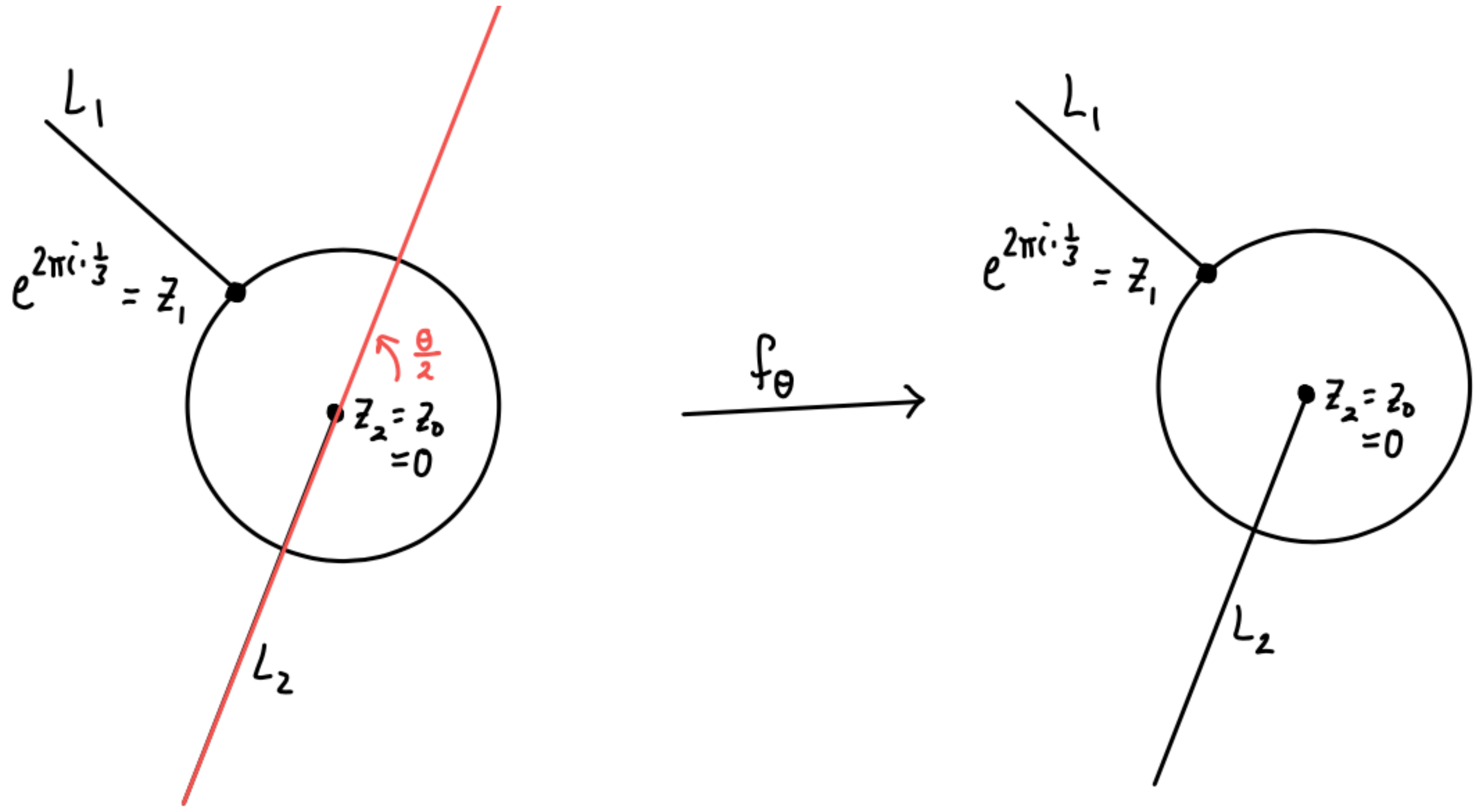


Thurston Spider Map: Quadratics

Build a Thurston map:

1. $f_\theta : \tilde{\mathbb{S}}_\theta \to \mathbb{S}_\theta$

$\theta = \frac{1}{3} \mapsto \frac{2}{3}$

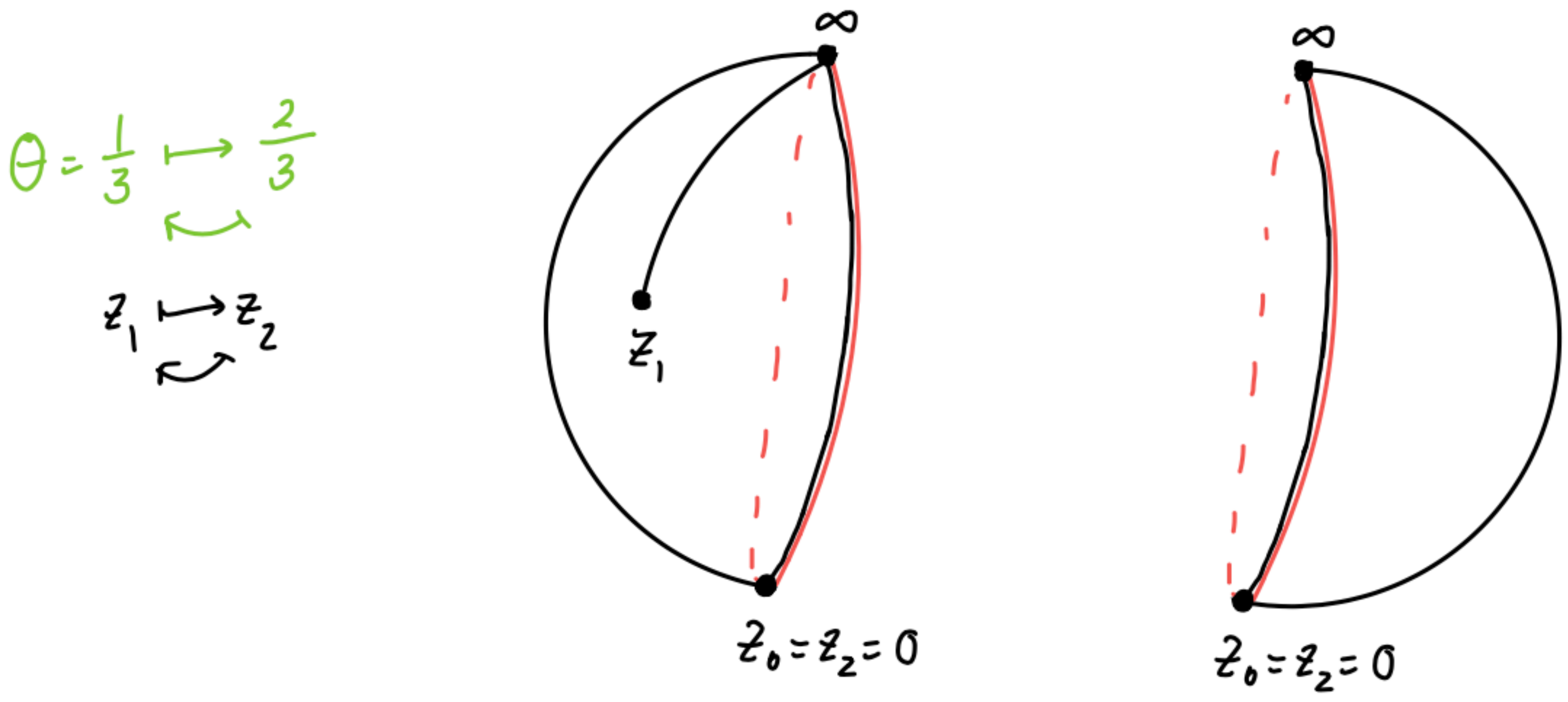


$$f_\theta(re^{it}) := \begin{cases} re^{2it} & \text{if } re^{it} \in \tilde{\mathbb{S}}_\theta \backslash (\{te^{2\pi i \cdot \theta/2} \mid t \in \mathbb{R}\} \cup L_{k-1}) \\ (r+1)e^{2it} & \text{if } re^{it} \in \{te^{2\pi i \cdot \theta/2} \mid t \in \mathbb{R}\}, r \geq 0 \\ (r-1)e^{2it} & \text{if } re^{it} \in L_{k-1}, r \geq 0 \end{cases}$$

Thurston Spider Map: Quadratics

Build a Thurston map:

2. View as halves on sphere



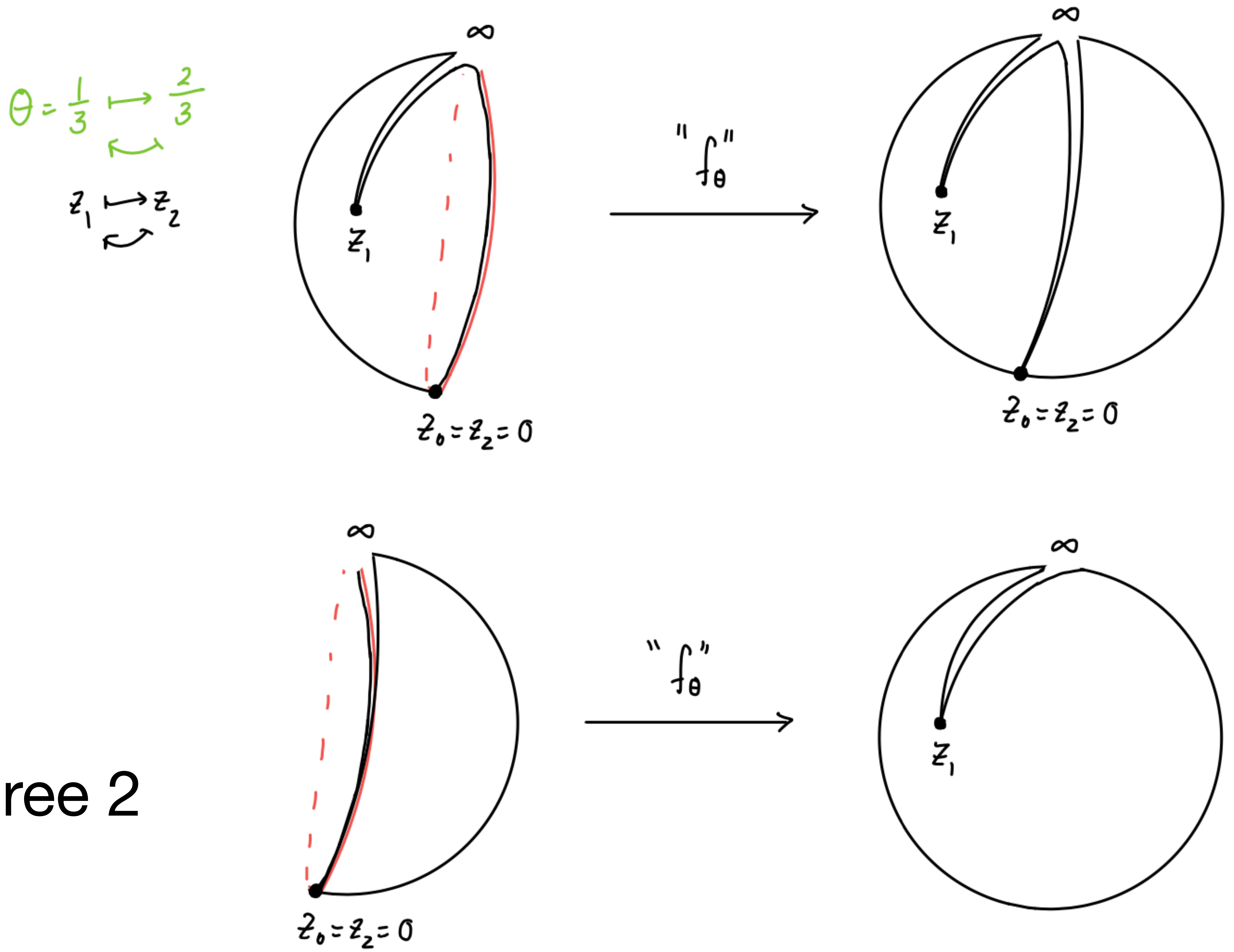
Thurston Spider Map: Quadratics

Build a Thurston map:

3. Cut legs and apply f_θ

4. Alexander trick + glue

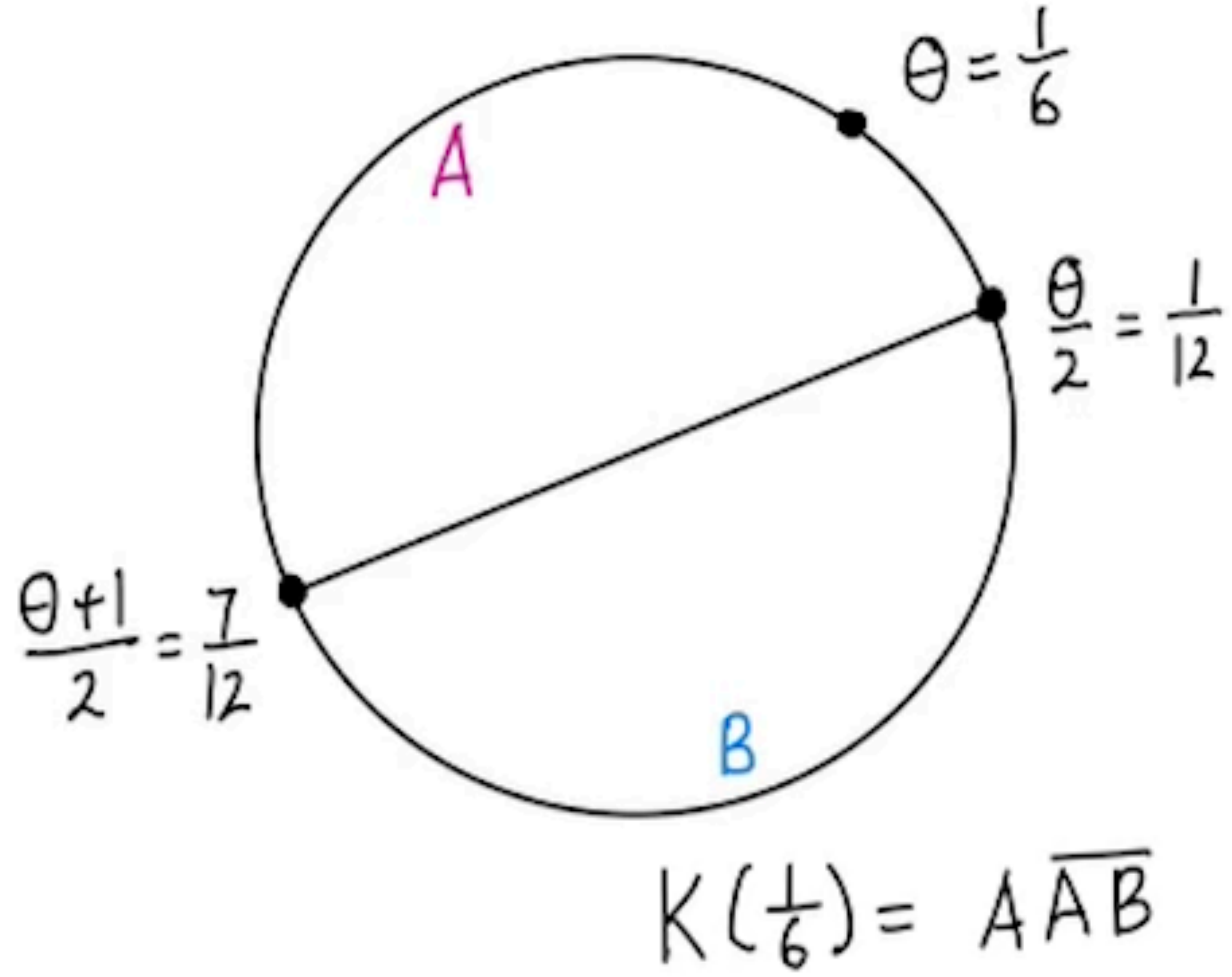
⟶ Thurston mapping $\tilde{f}_\theta : S^2 \to S^2$ of degree 2



$P_{\tilde{f}_\theta} = \{z_0, z_1\}$

Thurston Spider Map: Quadratics

Kneading Sequence: $K(\theta) := \{a_j\}$ s.t. $a_j = \begin{cases} A & 2^{j-1}\theta \in A \\ B & 2^{j-1}\theta \in B \end{cases}$

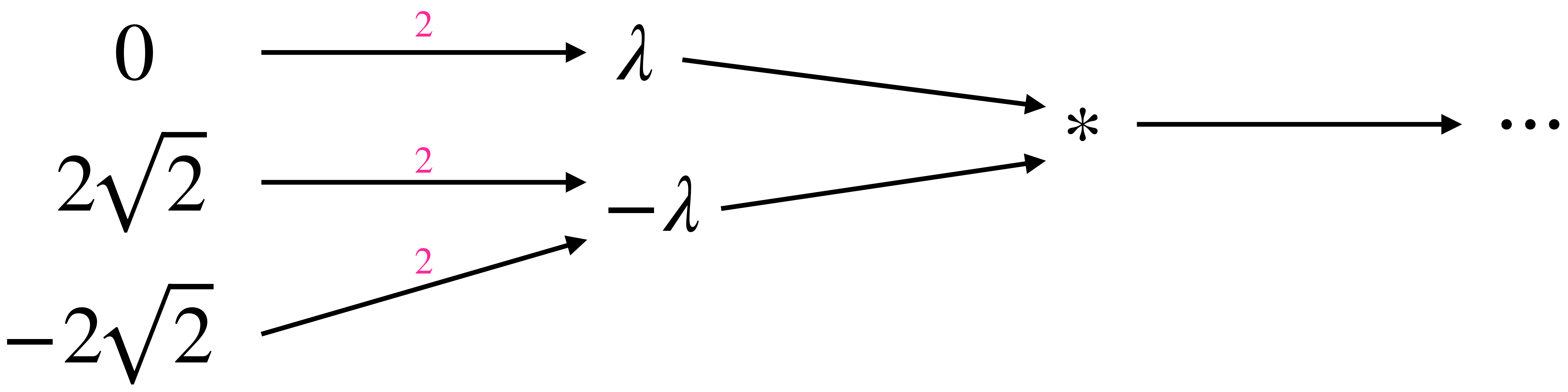


Theorem (Bielefeld, Fisher, Hubbard):

- θ periodic: $\tilde{f}_\theta : S^2 \to S^2$ has no Thurston obstruction.
- θ preperiodic: $\tilde{f}_\theta$ has no Thurston obstruction iff all distinct postcritical points have distinct kneading sequences.

Thurston Spider Map: Degree 4 Chebyshev

Want: Thurston spider map based off $f_{4,\lambda}(z) = \lambda T_4\left(\frac{z}{4}\right)$



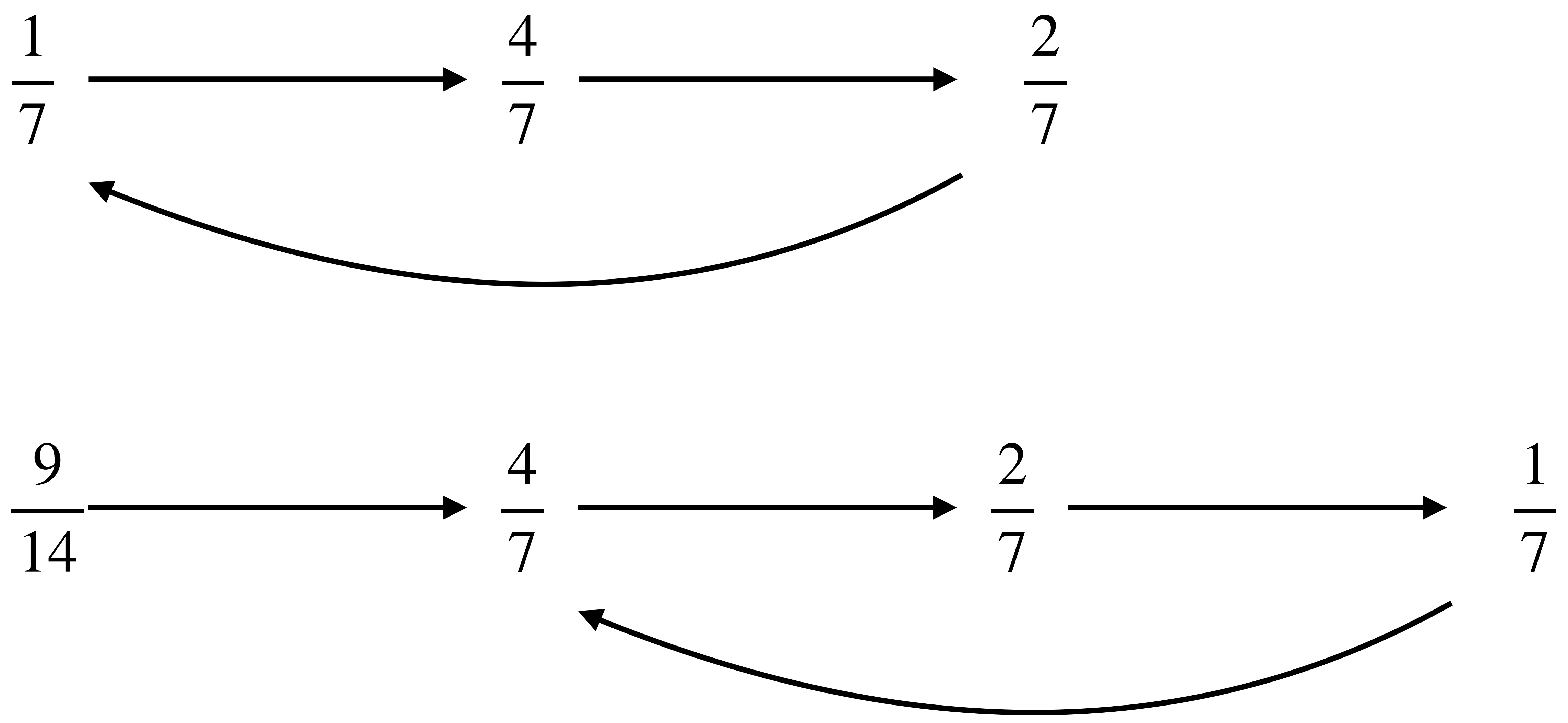
Given: $\theta \in \mathbb{Q}/\mathbb{Z}$.

$$\theta' := \theta + \frac{1}{2}.$$

1. θ periodic, θ' preperiodic
2. θ preperiodic, θ' periodic
3. θ preperiodic, θ' preperiodic

Thurston Spider Map: Degree 4 Example

Let $\theta = \dfrac{1}{7}, \quad \theta' = \dfrac{9}{14}.$



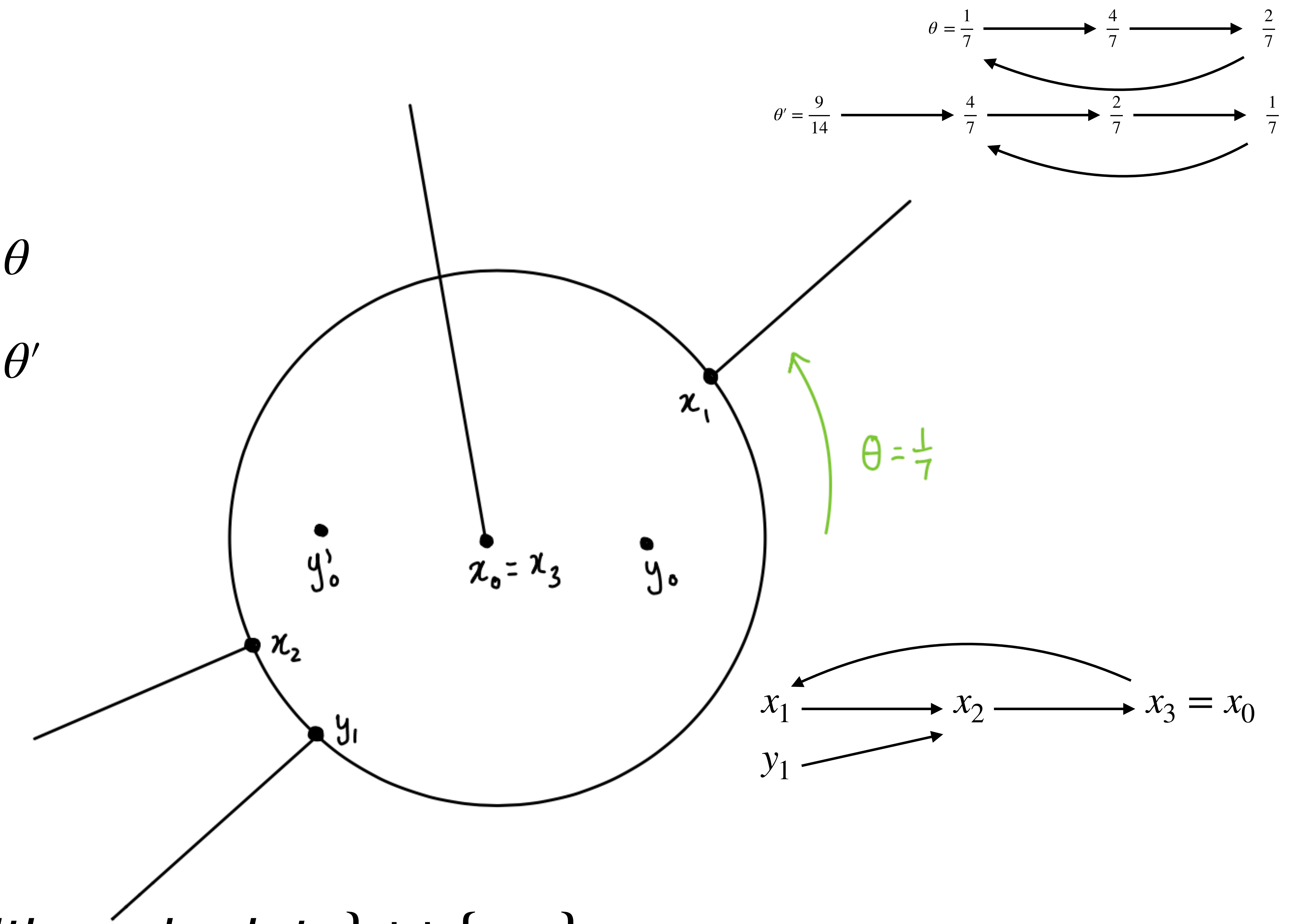
Thurston Spider Map: Degree 4 Example

Disk of radius 4:

$$x_0 = 0$$
$$y_0 = 2\sqrt{2}$$
$$y_0' = -2\sqrt{2}$$

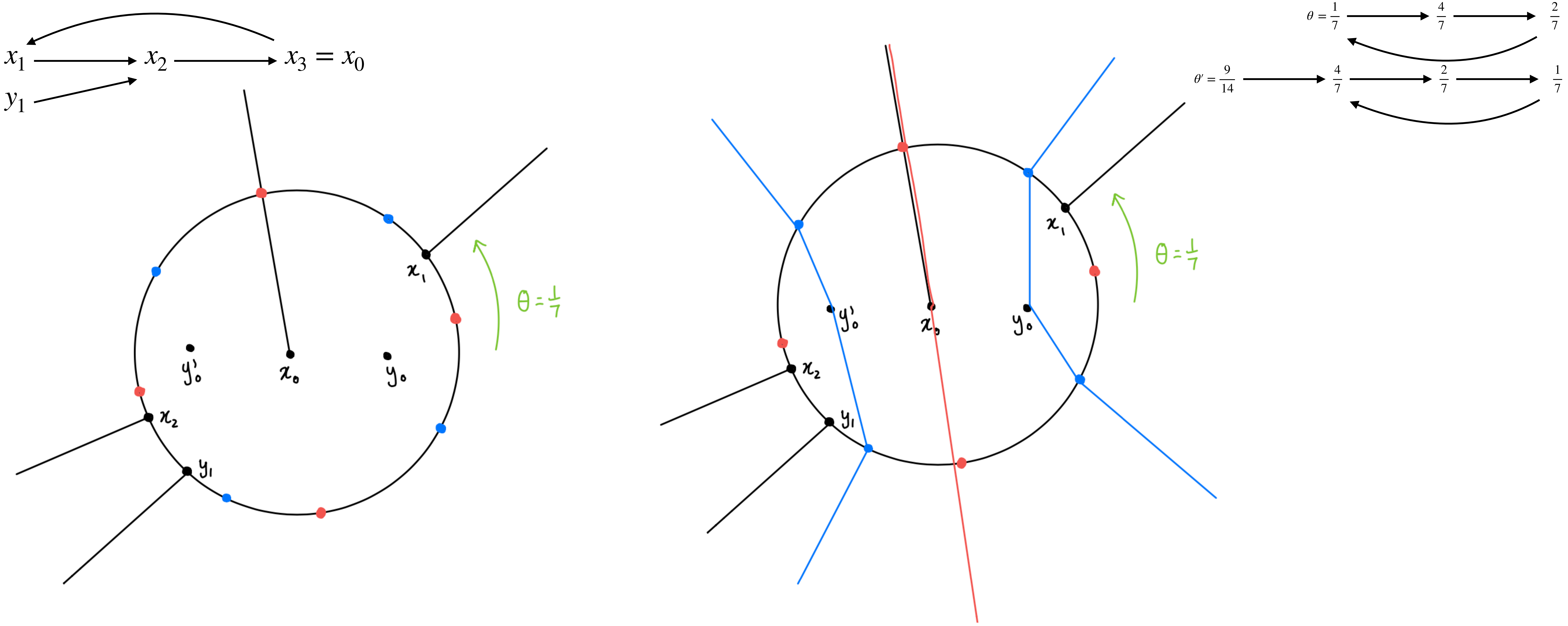
$j \geq 1$:

$$x_j = e^{2\pi i \cdot 4^{j-1}\theta}$$
$$y_j = e^{2\pi i \cdot 4^{j-1}\theta'}$$



(θ, θ')-Spider: $\mathbb{S}_{\theta,\theta'} :=$ {*legs with endpoints*} ∪ {∞}

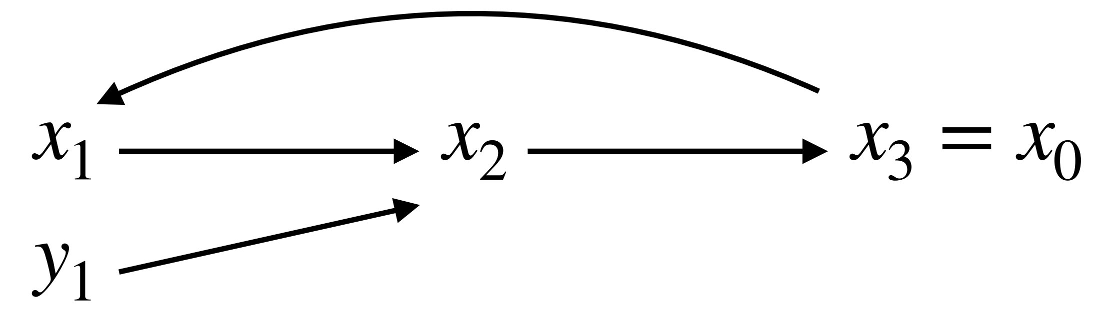
Thurston Spider Map: Degree 4 Example



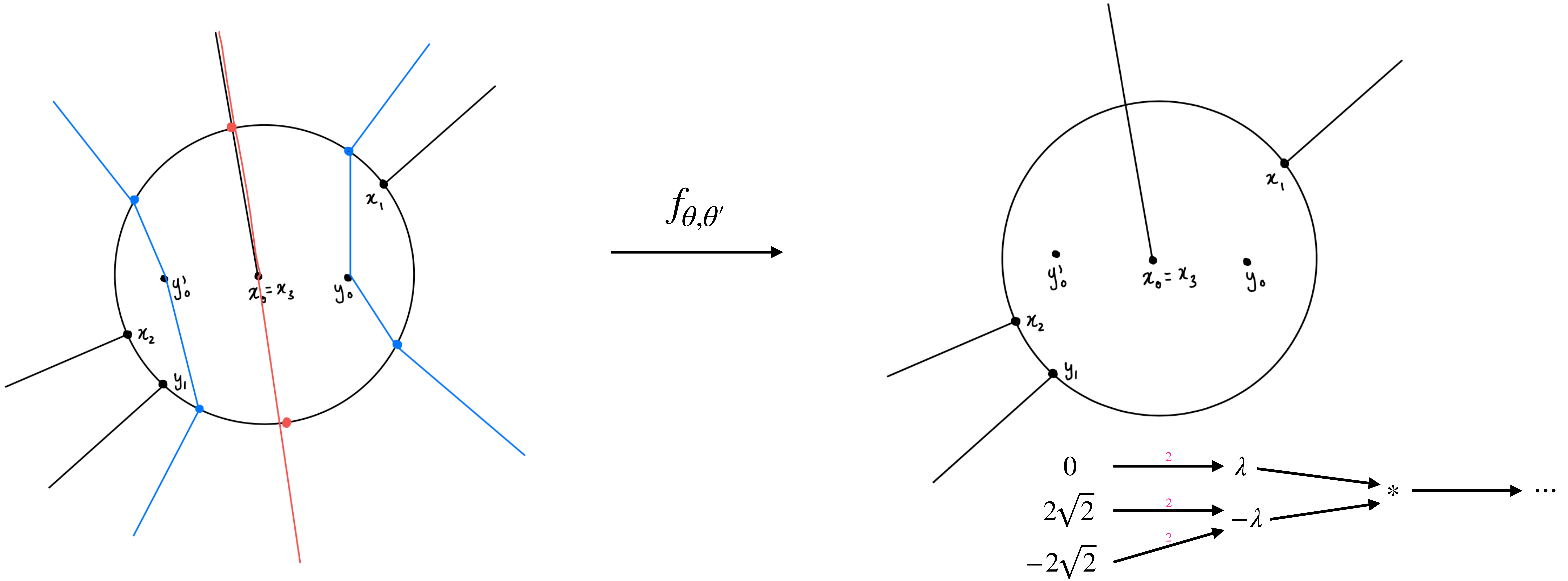
Extended (θ, θ')-*Spider:* $\tilde{\mathbb{S}}_{\theta,\theta'} := \mathbb{S}_{\theta,\theta'} \cup \{\textit{pink and blue lines}\}$

Thurston Spider Map: Degree 4 Example

Build a Thurston map:

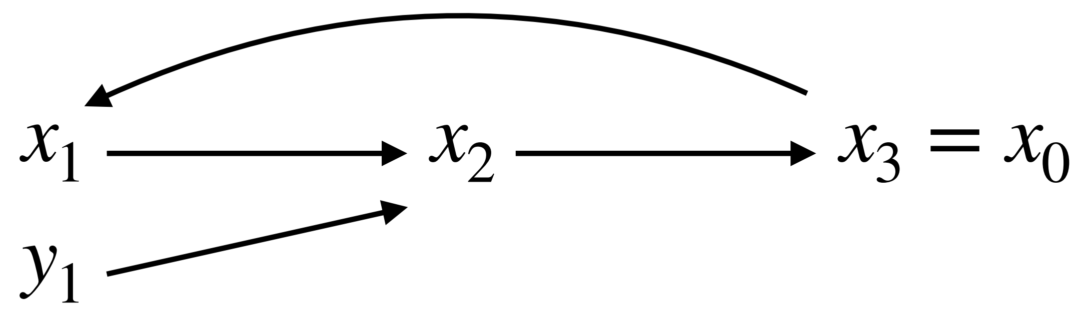


1. $f_{\theta,\theta'} : \tilde{\mathbb{S}}_{\theta,\theta'} \to \mathbb{S}_{\theta,\theta'}$

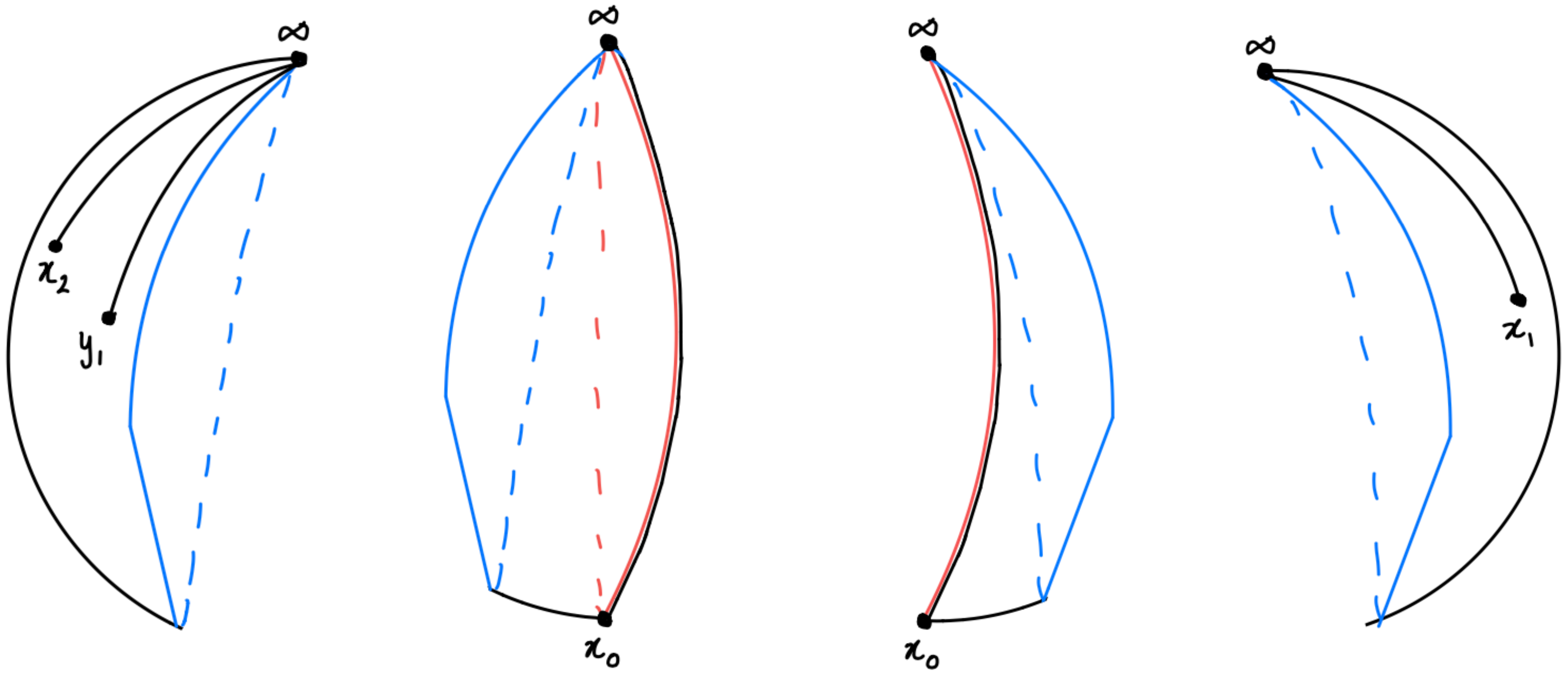


Thurston Spider Map: Degree 4 Example

Build a Thurston map:



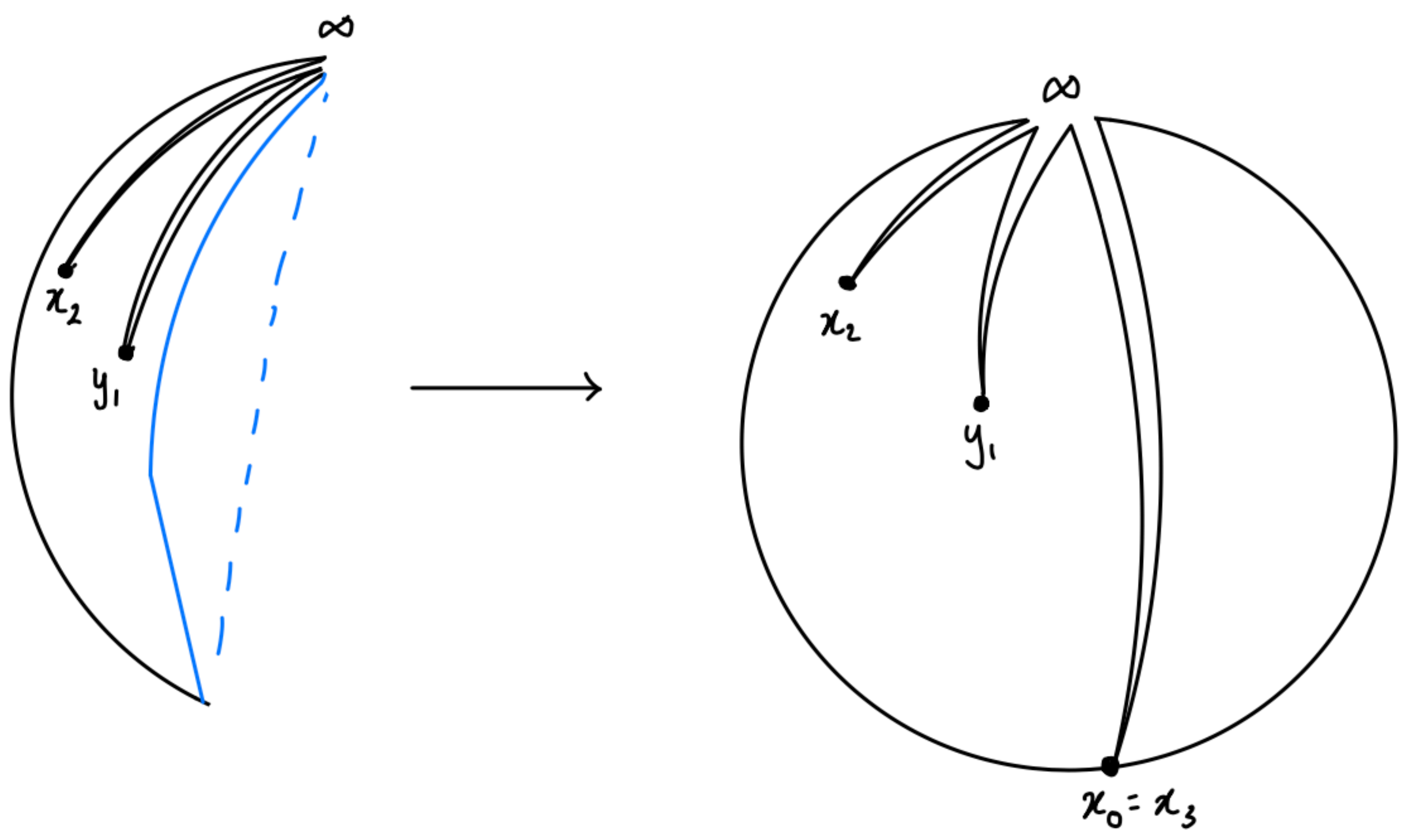
2. View as quarters on sphere

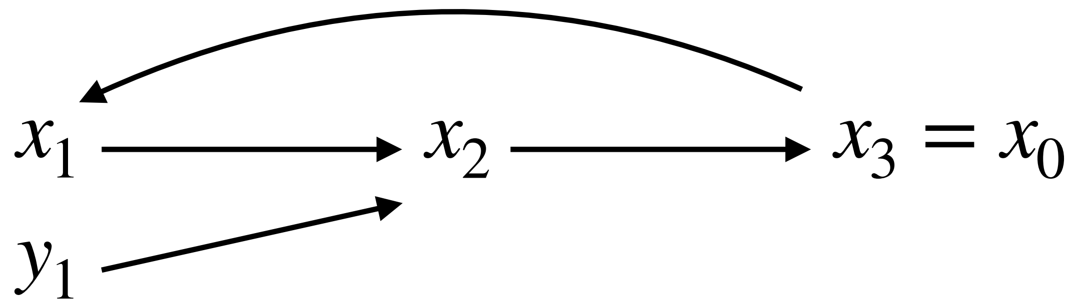


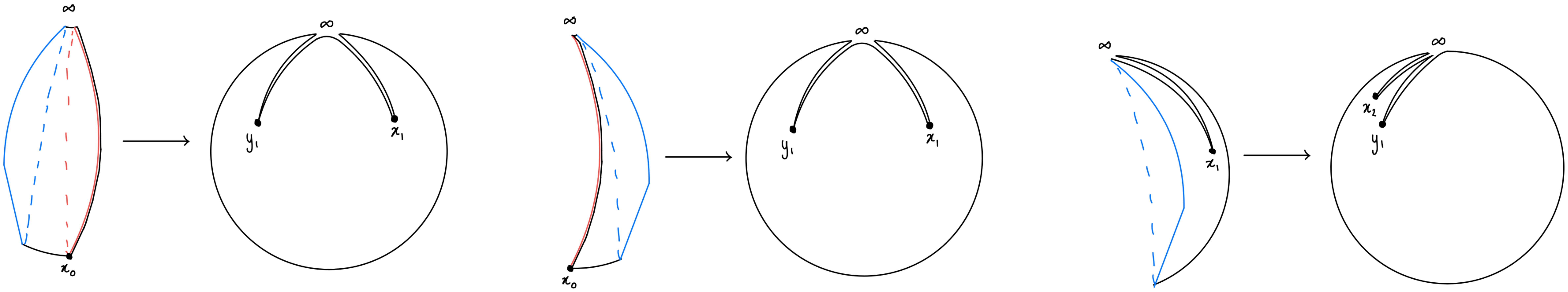
Thurston Spider Map: Degree 4 Example

Build a Thurston map:

3. Cut legs and apply $f_{\theta,\theta'}$

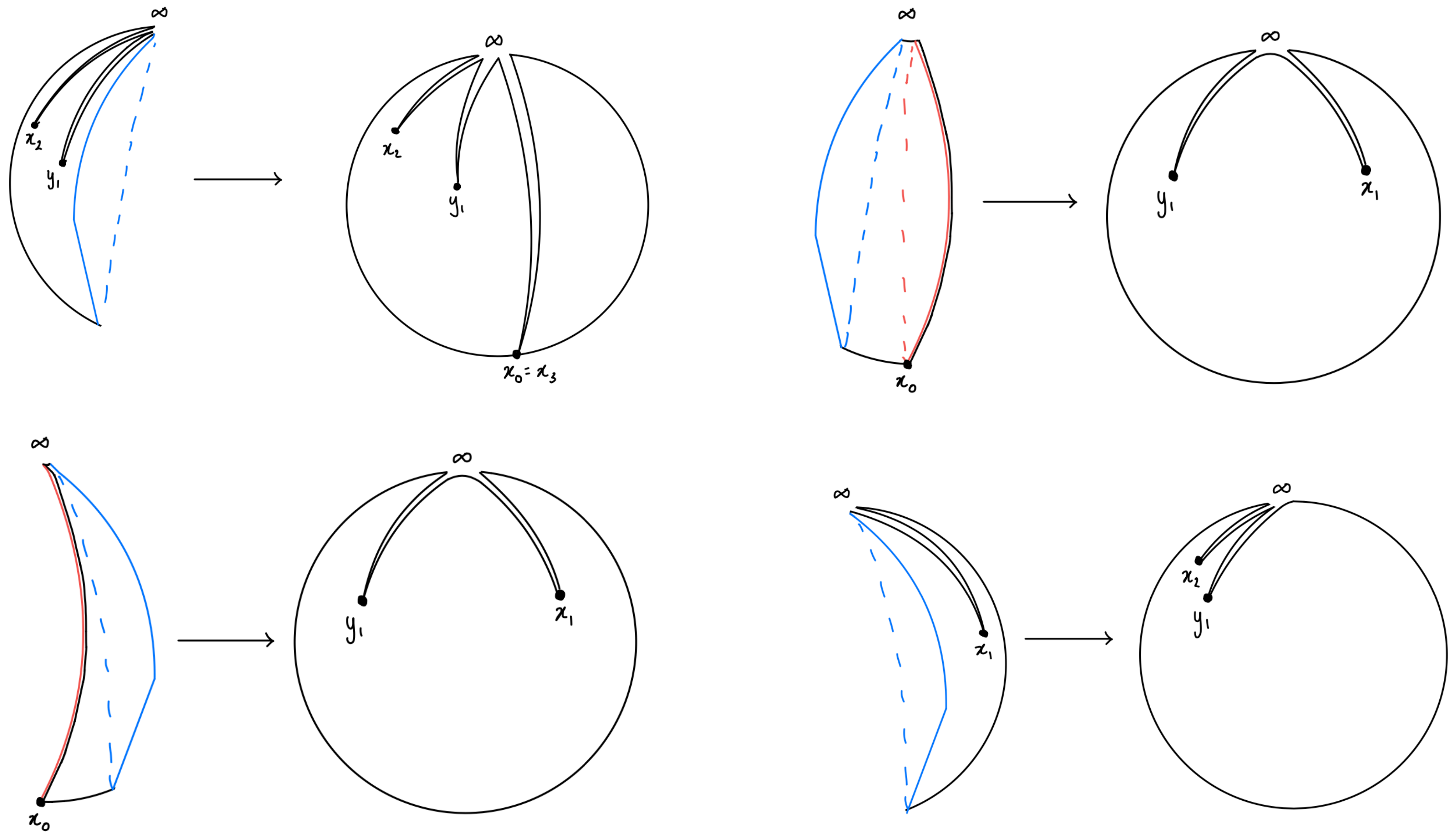






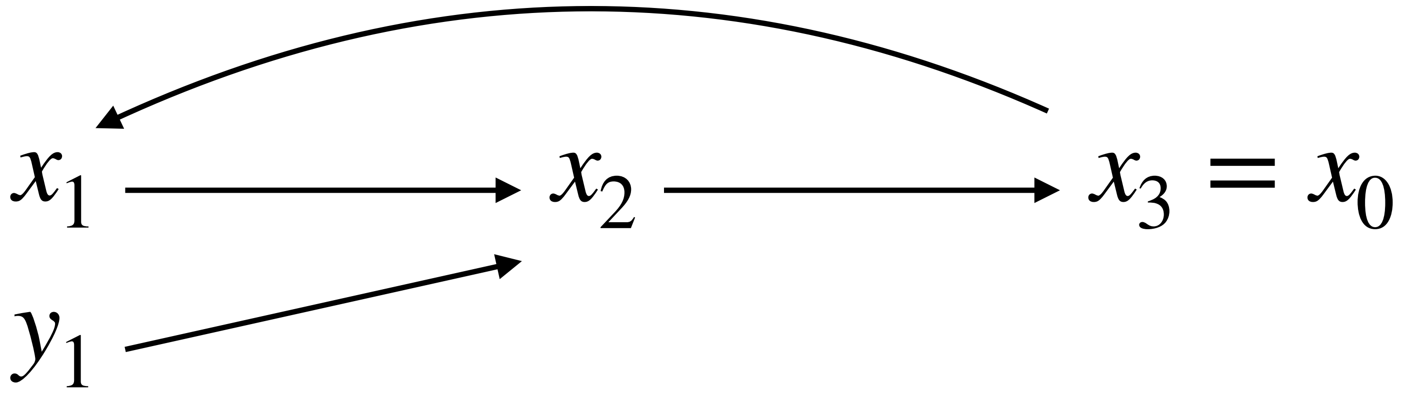
Thurston Spider Map: Degree 4 Example

4. Alexander trick



5. Put together maps from Step 4

$\longrightarrow$ Thurston mapping $\tilde{f}_{\theta,\theta'} : S^2 \to S^2$ of degree 4



Thurston Spider Map: Degree 4 Example

Kneading Sequence:

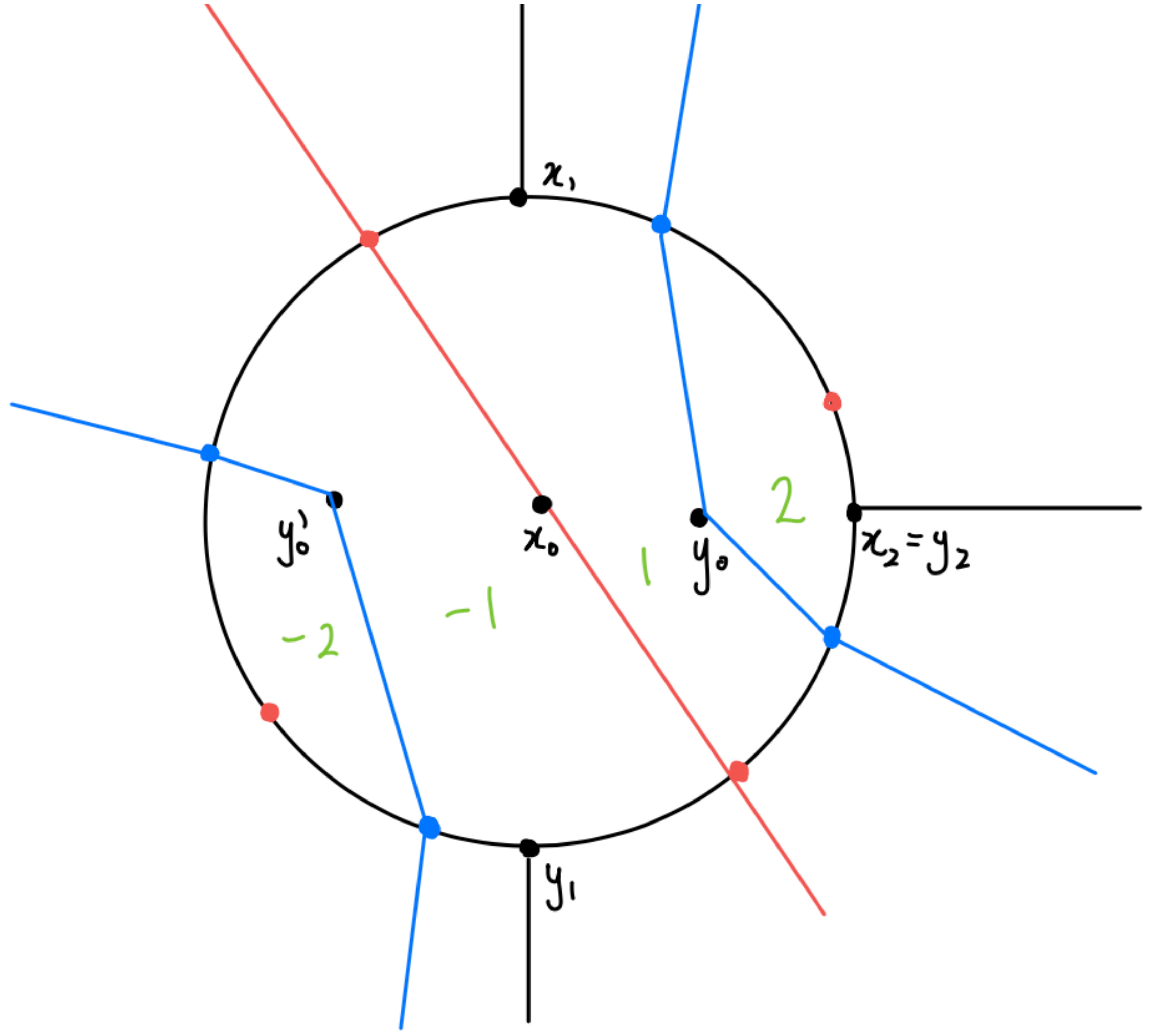
$$K(\theta) := \{a_j\} \text{ s.t. } a_j = \begin{cases} -2 & 4^{j-1}\theta \in \text{"}-2\text{"} \\ -1 & 4^{j-1}\theta \in \text{"}-1\text{"} \\ 1 & 4^{j-1}\theta \in \text{"}1\text{"} \\ 2 & 4^{j-1}\theta \in \text{"}2\text{"} \end{cases}$$

$\Theta = \frac{1}{4}$

$\Theta' = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$

$K(\frac{1}{4}) = 1\overline{2}$

$K(\frac{3}{4}) = -1\overline{2}$



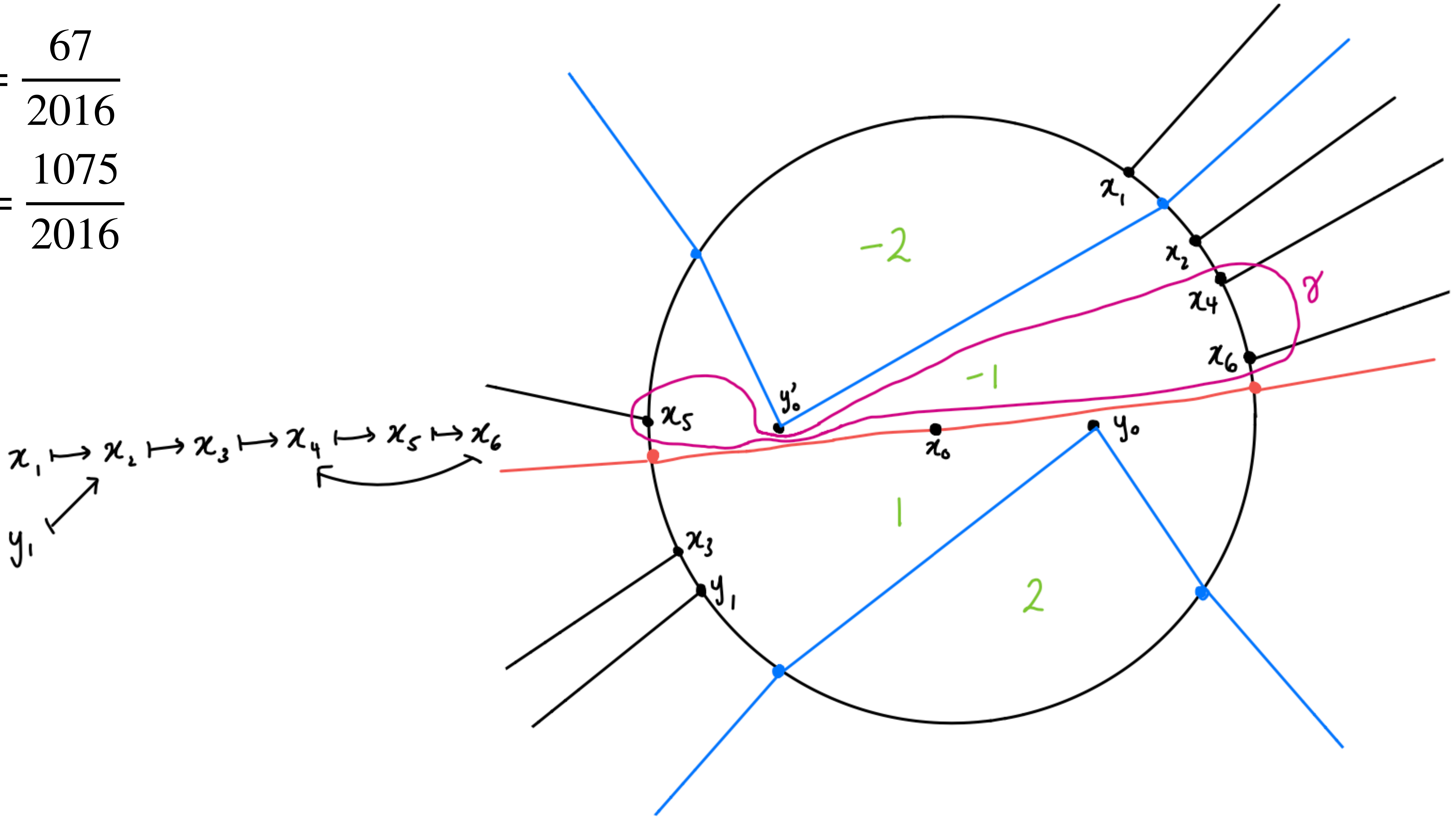
Theorem (D, 2023):

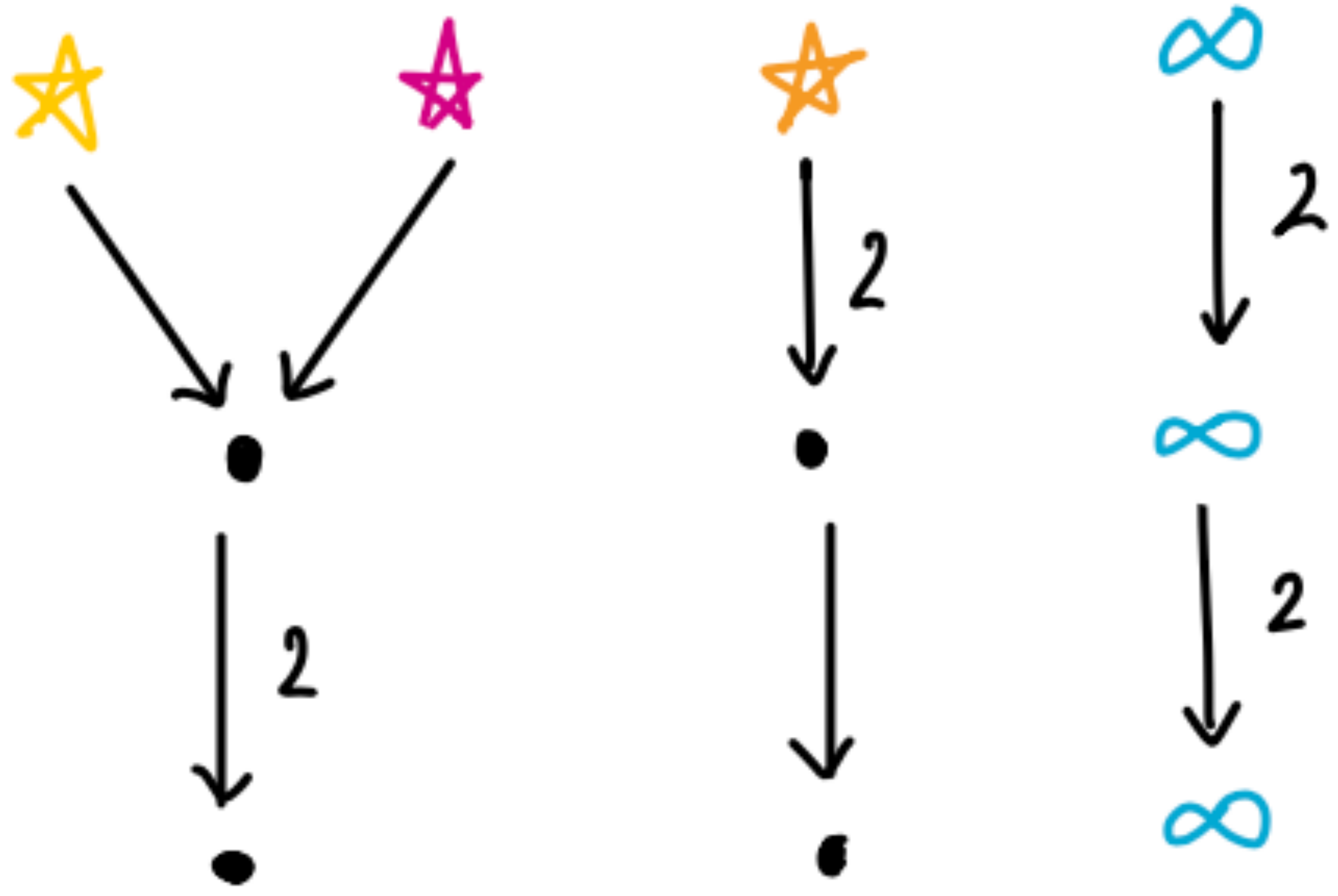
- θ or θ' periodic: $\tilde{f}_{\theta,\theta'} : S^2 \to S^2$ has no Thurston obstruction.
- θ and θ' preperiodic: $\tilde{f}_{\theta,\theta'}$ has no Thurston obstruction iff all distinct postcritical points have distinct kneading sequences.

Thurston Obstruction Example

$$\theta = \frac{67}{2016}$$

$$\theta' = \frac{1075}{2016}$$

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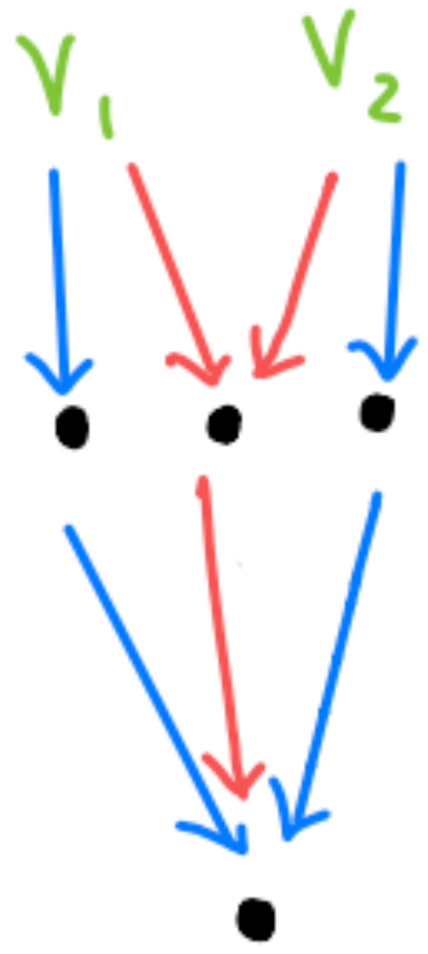
V1
V2

Future Directions/Questions

$$f_{2n,\lambda}(z) = \lambda(-1)^n T_{2n}\left(\frac{z}{2n}\right)$$

- Thurston equivalence of $\tilde{f}_{\theta,\theta'}$ to $f_{4,\lambda}$
- Understand Thurston pullback map
- Convergence of Thurston pullback maps