Transcendental Wandering Triangles

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joint work in progress with Jordi Canela and Lasse Rempe

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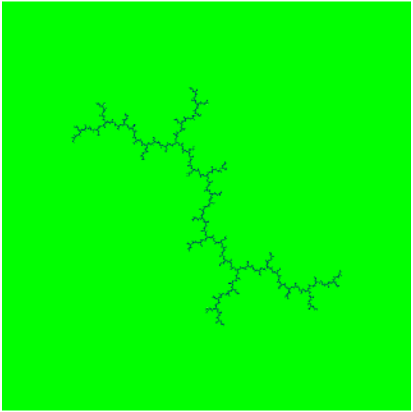
Topics in Complex Dynamics 2023, Barcelona

Thurston's no wandering triangles theorem

Theorem (Thurston 1985)

branch point of a locally connected Julia set of a quadratic polynomial is eventually periodic or precritical.



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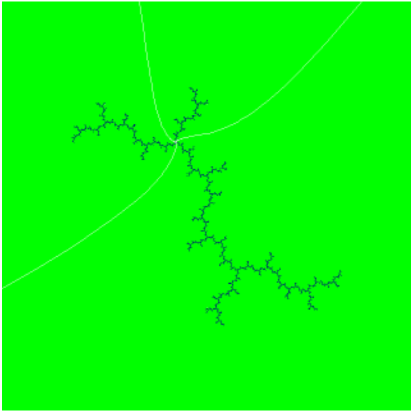

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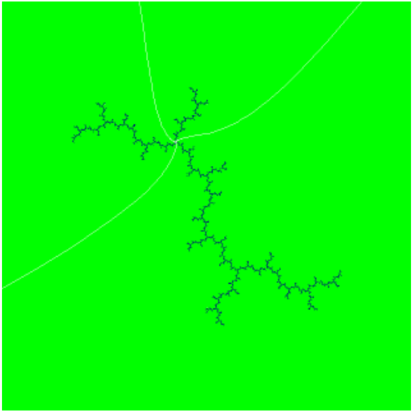

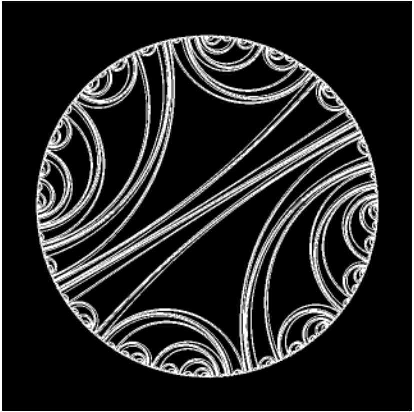

Wandering triangles for polynomials

Theorem (Blokh 2005)

If a cubic polynomial has a wandering non-precritical branching point, then the two critical points are recurrent to each other.

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Theorem (Blokh–Oversteegen 2008)

There exist cubic polynomials with wandering non-precritical branching points.

lternative proof by Buff–Canela–Roesch via pertubation of postcritically finite polynomials.

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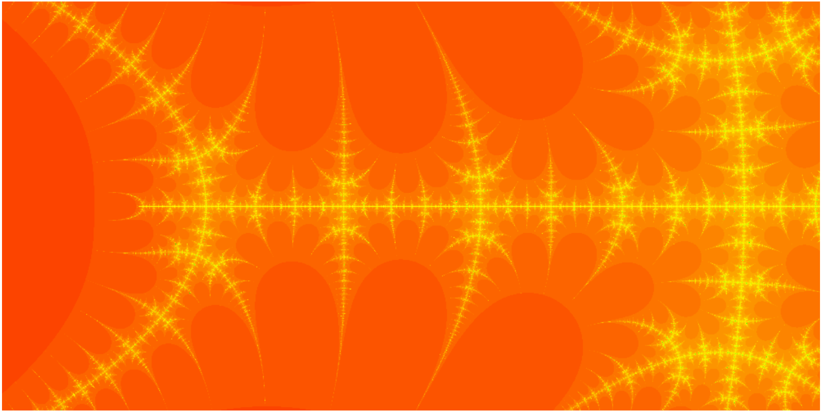

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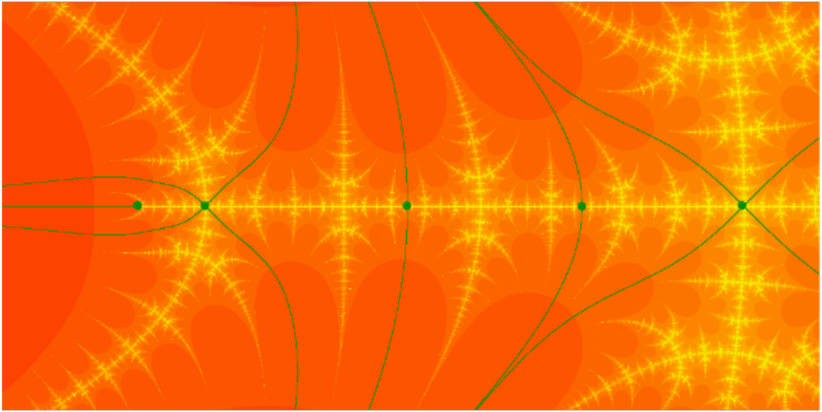
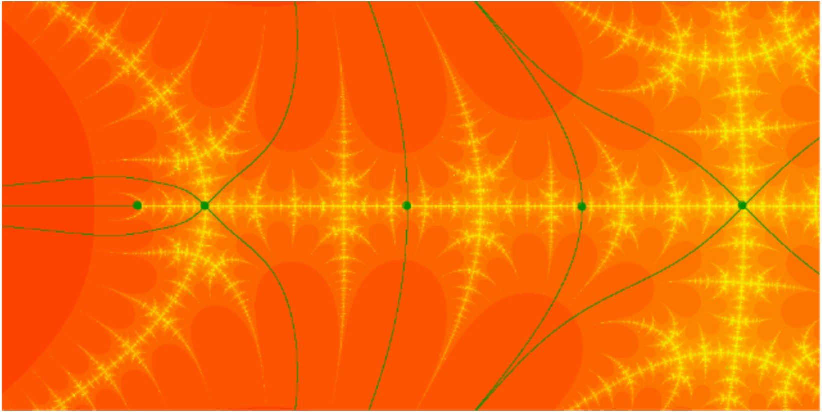


Solution: use combinatorics of $I(f) = \{z \in \quad : f^n(z) \to \infty\}$

dynamic ray is a maximal injective curve $\gamma\colon (0,\infty) \to I(f)$ with

- $f^n(\gamma(t)) \to \infty$ as $t \to \infty$ for all $n \geq 0$
- $f^n(\gamma(t)) \to \infty$ as $n \to \infty$ uniformly on $[\epsilon, \infty)$ for all $\epsilon > 0$

The **endpoint** / landing point of a ray is $\lim_{t\to 0} \gamma(t)$ if it exists.



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Theorem (Rottenfußer–Rückert–Rempe–Schleicher 2011)

Let f be a post-singularly bounded entire function of finite order. Then $I(f)$ consists entirely of dynamic rays (possibly with endpoints).

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Theorem (lhabib–Rempe 2017)

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Work-In-Progress (Canela–R.–Rempe)

It is possible for the family $a\cos(\sqrt{z}) + b$.

The family $\cos(\sqrt{z}) + b$

We consider the family $f_{\ b}(z) = a\cos(\sqrt{z}) + b$

- critical points $k^2\pi^2$ for $k \geq 1$
- critical values $a + b$ for k even, $\quad a + b$ for k odd
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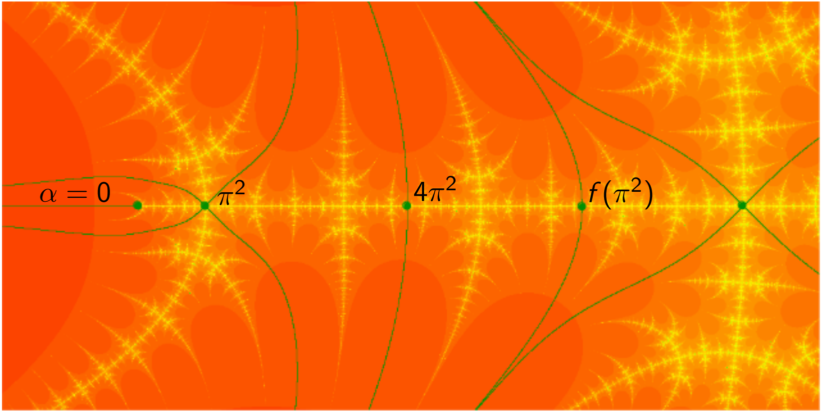
If $f_{\ b}$ is post-singularly finite, then it is strongly subhyperbolic, so it is docile (Mihaljevic-Brandt/ lhamed–Rempe–Sixsmith).
In particular for $f_{\ b}$ post-singularly finite we have

- bounded dynamic rays land
- repelling periodic points are landing points of dynamic rays

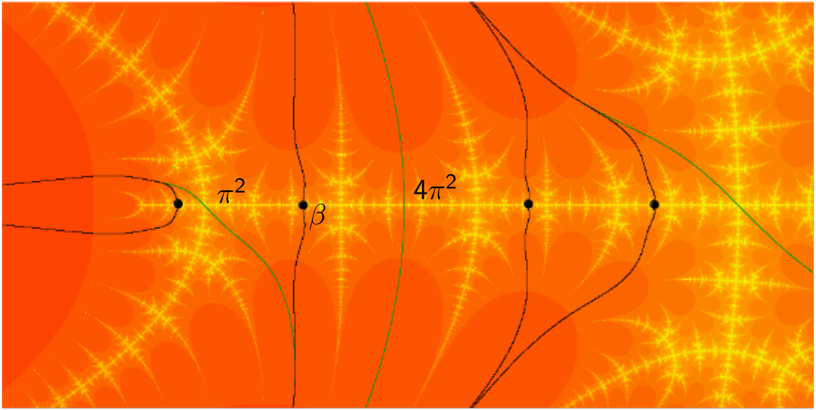
Proof strategy

Start with $f_{\ 0\ b_0}$ real post-singularly finite:

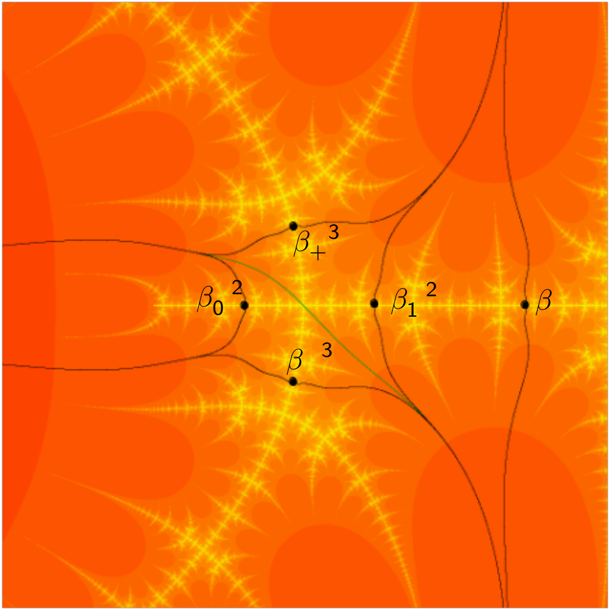


$$b_0 = \quad a_0 \approx 32.55,\ f^2(\pi^2) = 4\pi^2,\ f((2k)^2\pi^2) = 0$$

Rays for repelling fixed point

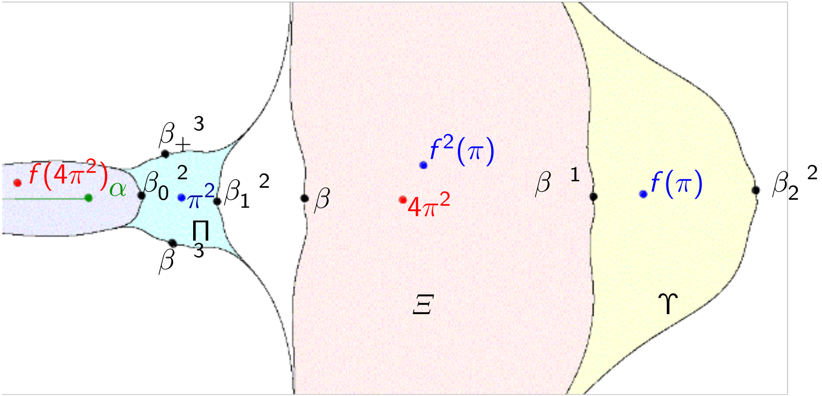


Iterated preimages of



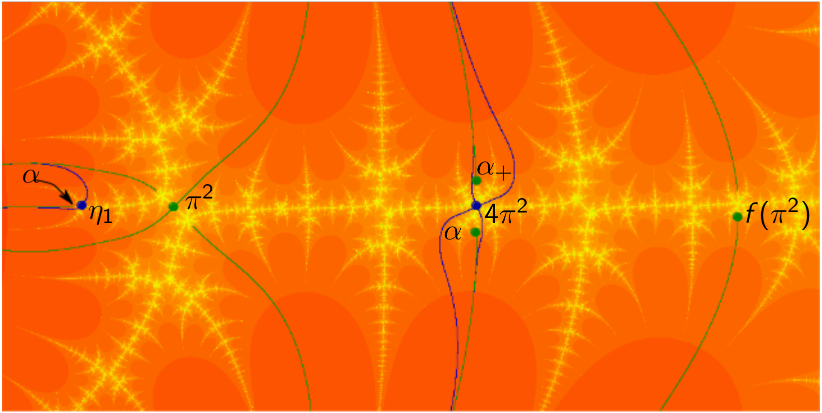
dmissible maps

Pertubations of $f_{0\ b_0}$ that preserve the figure below are **dmissible**.



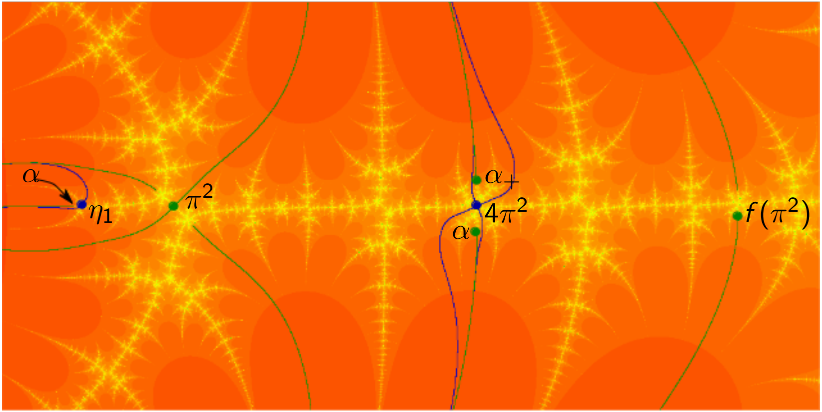
n admissible map f has a (k,ℓ)-*configuration* if the critical points $w, w' \in \{\pi^2, 4\pi^2\}$ satisfy $f^k(w) = w'$ and $f^\ell(w') = \alpha$.

Ide : we perturb f iteratively interchanging the roles of $\pi^2, 4\pi^2$



Construction of wandering triangle

Control ξ landing point of rays seperating α, α and α_+



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Thank you!