Dynamical approximations of postsingularly finite entire maps

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Joint work with Malavika Mukundan and Bernhard Reinke

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Notations and conventions

If f is an entire map, then

- S_f is a set of singular values of f;
- f is of finite type or belongs to class $\mathcal{S}$ if S_f is finite;

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- postsingular set of f is

$$P_f := \bigcup_{n \geqslant 0} f^{\circ n}(S_f);$$

- f is postsingularly finite if P_f is finite.

Example

$$f(z) = c(1 - \exp(z^2)), \text{ where } c = \sqrt{\ln 2}$$

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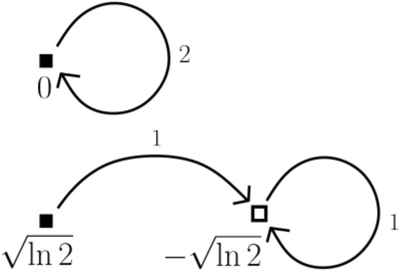

Dynamical approximations

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Further works by Fagella, Kisaka, Krauskopf, Kriete, Mihalević-Brandt, Morosawa...

Main Theorem

Theorem (Mukundan - NP - Reinke'23).

Let f be a postsingularly finite entire map, then there exists sequence of postsingulalry finite polynomials (p_n) such that

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Maps of finite type and graphs

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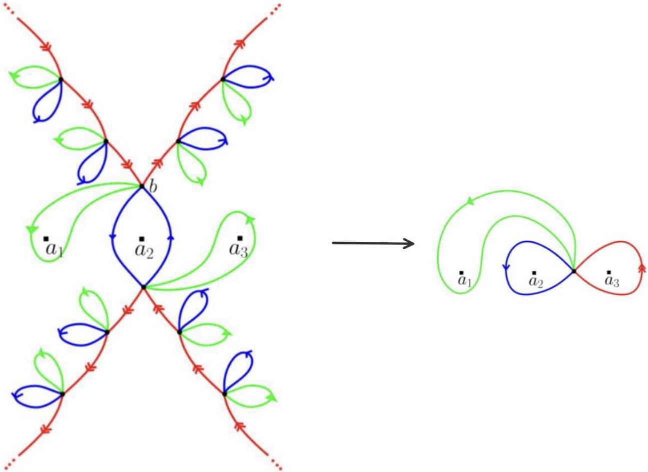
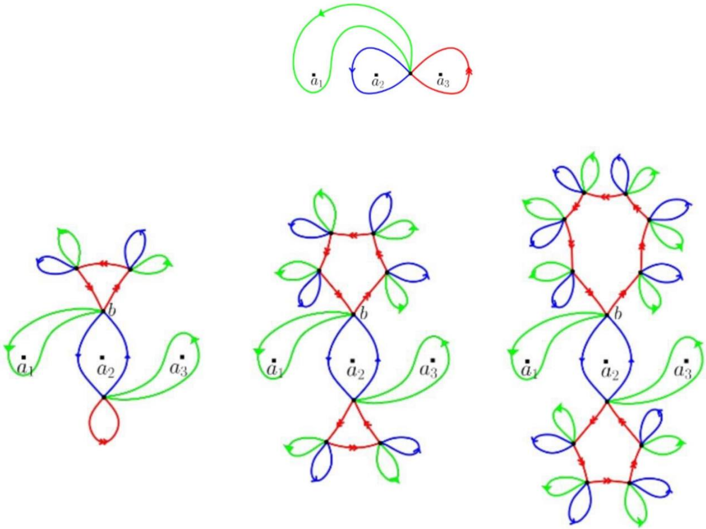


Sequence of graphs



More elaborated formulation

$f\colon (R^2, A) \circlearrowleft$ and $f_n\colon (R^2, A) \circlearrowleft, n \in \mathbb{N}$ are Thurston maps and (f_n) converges combinatorially to f.

Theorem (Mukundan - NP - Reinke'23).

The sequence (σ_{f_n}) converges to σ_f locally uniformly on $\mathrm{Teich}(R^2, P_f)$.

Theorem (Mukundan - NP - Reinke'23).

If g is a postsingularly finite entire map Thurston equivalent to f, then there exists a sequence of postsingularly finite entire maps (g_n) converging locally uniformly to g, where g_n is Thurston equivalent to the map f_n for sufficiently large n.

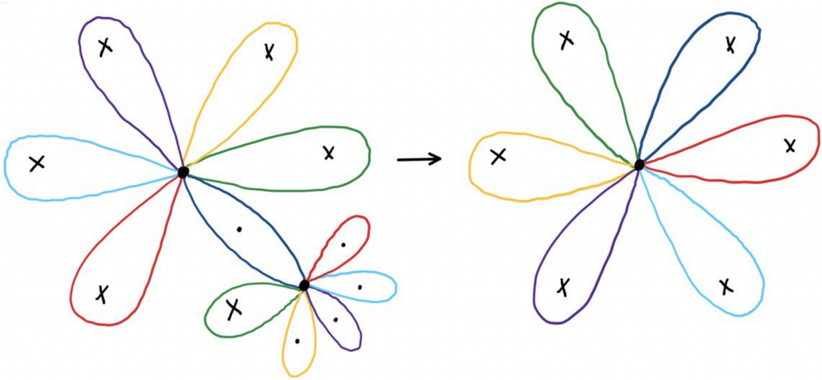
When combinatorial/topological model is realized by holomorphic map?

State of art

Family of maps	Authors
Topological polynomials and branched covering	W.Thurston'80s, Douady-Hubbard'93
Topological multi-error maps and their compositions	Hubbard-Schleicher-Shishikura'09, S. Shemyakov'22
New families of maps having infinitely many asymptotic tracts and no critical points	NP'23

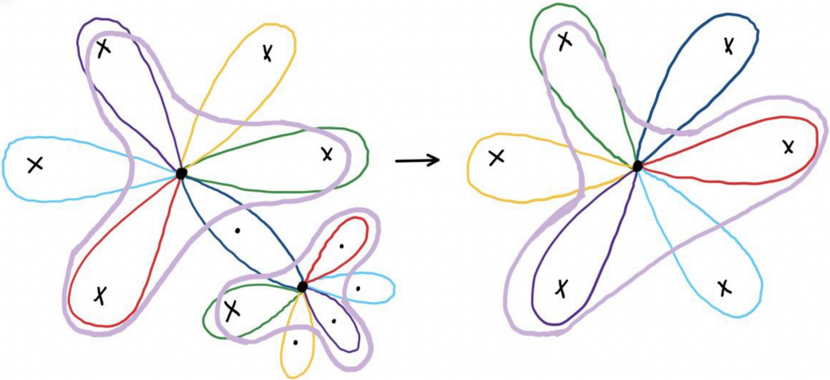
Moltes gràcies!

One more example



$$x_1 \longrightarrow x_2 \longrightarrow x_3 \longrightarrow x_4 \longrightarrow x_5 \longrightarrow x_6$$

Example of obstruction



Definition of topologically holomorphic map

Definition.

We say that map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is topologically holomorphic if for every $x \in \mathbb{R}^2$ there exists $k \in \mathbb{N}$, an open neighbourhood U and two orientation-preserving homeomorphisms $\psi\colon U \to \mathbb{D}$ and $\varphi\colon f(U) \to \mathbb{D}$ such that $\psi(x) = 0$, $\varphi(f(x)) = 0$ such that the diagram commutes

$$\begin{array}{ccc} U & \xrightarrow{f|U} & f(U) \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\varphi} \\ \mathbb{D} & \xrightarrow{z \mapsto z^k} & \mathbb{D} \end{array}$$