Newton-like components in the Chebyshev-Halley family of degree n polynomials

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Dynamics of holomorphic maps on the Riemann sphere

The **Riemann sphere** is the set $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.
Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a holomorphic map. Then, R is a rational map of the form

$$R(z) = \frac{p(z)}{q(z)},$$

where p and q are coprime polynomials such that at least one has degree greater or equal than 2. We also denote by

$$R^n := \underbrace{R \circ R \circ \cdots \circ R}_{\text{n times}},$$

the n-th iterate of R.

Definition

Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map. The set of points $z \in \hat{\mathbb{C}}$ such that there exists U a neighbourhood of z such that $\{R^n|_U\}$ is normal, is called the **Fatou set** $\mathcal{F}(R)$. The complement of the Fatou set is called the **Julia set** $\mathcal{J}(R)$.

Definition

A connected component of the Fatou set is called a **Fatou component**.

Periodic Fatou components

Definition

Let R be a rational map. A Fatou component U is **preperiodic** if there exist $n \geq 0$ and $p > 0$ such that $R^{n+p}(U) = R^n(U)$ (in particular, if $n = 0$, we say that U is **periodic**).

Theorem (No Wandering Domains Theorem. Sullivan, 1983)

Any Fatou component of a rational map is preperiodic.

Theorem

Periodic Fatou components of rational maps can have connectivity 1, 2 *or* ∞.

Newton's method

Definition

For a differentiable real function g, **Newton's method** is defined as follows:

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}.$$

Theorem (Shishikura, 1991)

Let f be a polynomial, and N_f its associated Newton map. Then, $\mathcal{J}(N_f)$ is connected.

Lemma (Tan Lei, 1997)

Any rational map F of degree d having d distinct superattracting fixed points is conjugate by a Möbius map to N_P, where P is a polynomial of degree d. Moreover, if $z = \infty$ is a fixed point, but not superattracting, then $F = N_P$.

The Chebyshev-Halley methods

Definition

For a holomorphic function f, the **Chebyshev-Halley method** is defined as follows:

$$z_{n+1} = z_n - \left(1 + \frac{1}{2}\frac{L_f(z_n)}{1 - \alpha L_f(z_n)}\right)\frac{f(z_n)}{f'(z_n)},$$

where $\alpha \in \mathbb{C}$ and $L_f(z) = \frac{f(z)f''(z)}{(f'(z))^2}$.

The family $O_{n,\alpha}$

The family $O_{n,\alpha,c}$ is obtained by applying the Chebyshev-Halley family of methods to the family $P_c(z) = z^n + c$, $c \in \mathbb{C}$. For any $c \in \mathbb{C}^*$, the maps $O_{n,\alpha,c}$ and $O_{n,\alpha} := O_{n,\alpha,-1}$ are conjugated. Thus, it suffices to study:

$$O_{n,\alpha}(z) = \frac{(1-2\alpha)(n-1) + (2-4\alpha-4n+6\alpha n-2\alpha n^2)z^n + (n-1)(1-2\alpha-2n+2\alpha n)z^{2n}}{2nz^{n-1}(\alpha(1-n)+(-\alpha-n+\alpha n)z^n)}.$$

Lemma (Symmetry with respect to rotation by a root of unity. Campos-Canela-Vindel, 2020)

Let $n \in \mathbb{N}$. Let ξ be an n-th root of the unity, i.e. $\xi^n = 1$ and $I_\xi(z) = \xi z$. Then

$$O_{n,\alpha} \circ I_\xi(z) = I_\xi \circ O_{n,\alpha}(z).$$

The dynamical plane of $O_{n,\alpha}$, for $n = 3$ and $\alpha = 1.7 - 1.8i$.

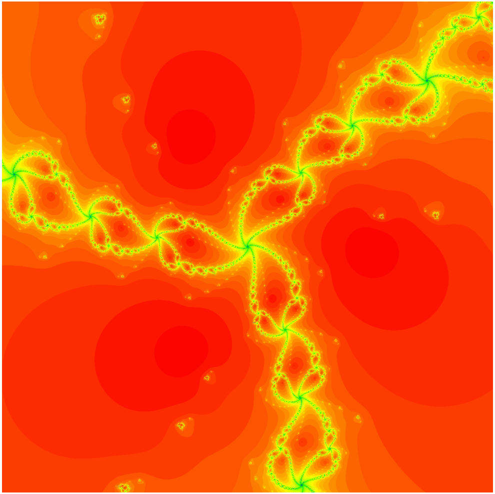

Figure: The dynamical plane of $O_{n,\alpha}$, for $n = 3$ and $\alpha = 1.7 - 1.8i$

Properties of the map $O_{n,\alpha}$

Lemma (Campos-Canela-Vindel, 2020)

Let $n \geq 2$ and $\xi \in \mathbb{C}$, such that $\xi^n = 1$. For all $\alpha \in \mathbb{C} \setminus \{\frac{1}{2}, \frac{2n-1}{2n-2}\}$, the basin of attraction $A_{n,\alpha,\xi}$ contains at most one critical point other than $z = 1$.

Theorem (Campos-Canela-Vindel, 2020)

For fixed $n \geq 2$ and $\alpha \in \mathbb{C}$, the immediate basins of attraction of the roots of $z^n - 1$ under $O_{n,\alpha}$ are multiply connected if and only if $A^*_{n,\alpha}(1)$ contains a critical point $c \neq 1$ and no preimage of $z = 1$ other than $z = 1$ itself.

Main results

Theorem A

Fix $n \geq 2$ and assume that $A^*_{n,\alpha}(1)$ is infinitely connected for some $\alpha \in \mathbb{C}$. Then there exists an invariant Julia component Π (which contains $z = \infty$) which is a quasiconformal copy of the Julia set of N_{f_n}, where N_{f_n} is the map obtained by applying Newton's method to the polynomial $f_n(z) = z^n - 1$.

Theorem B

Let $n \geq 2$. Then there exists $\alpha_0 > 0$ large enough such that for $\alpha > \alpha_0$, $\alpha \in \mathbb{R}$, $A^*_{n,\alpha}(1)$ is infinitely connected. Moreover, for $n = 2$, the statement is true for any $\alpha \in \mathbb{C}$ such that $|\alpha| > \alpha_0$.

Newton and Chebyshev-Halley dynamical planes

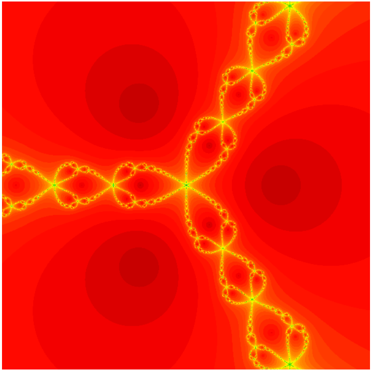

Figure: The dynamical plane of N_{f_3}, where $f_3(z) = z^3 - 1$

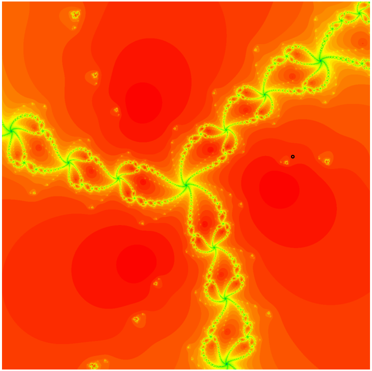

Figure: Dynamical plane of $O_{n,\alpha}$, for $n = 3$ and $\alpha = 1.7 - 1.8i$

Description of the immediate basin of attraction of $z = 1$, $A^*_{n,\alpha}(1)$

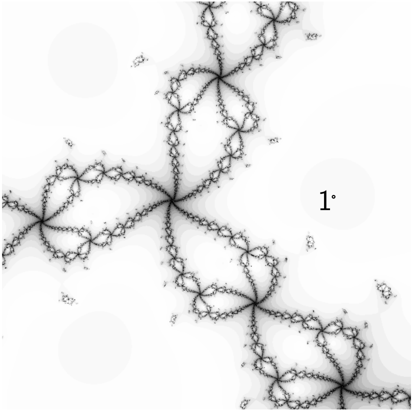

Figure: $A^*_{n,\alpha}(1)$ for $n = 3$ and $\alpha = 0.2 + 1.592i$

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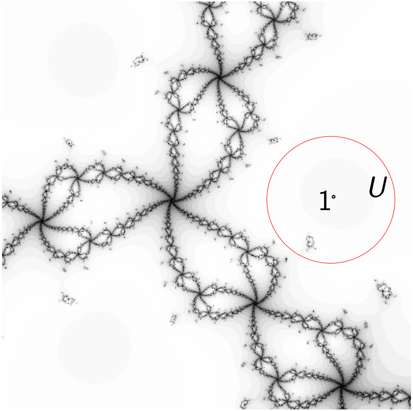

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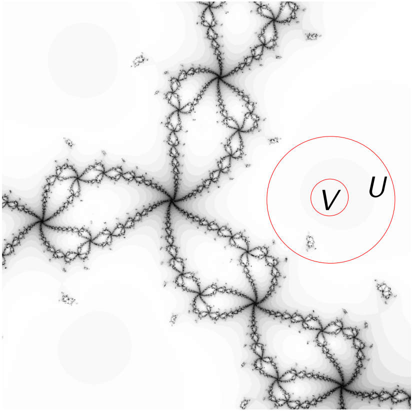

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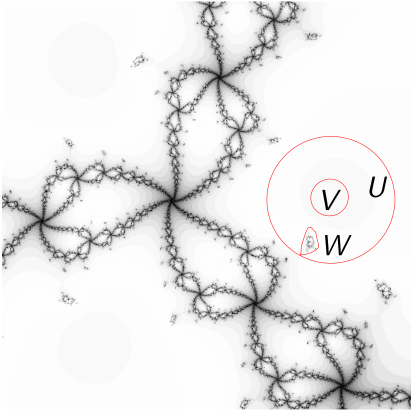

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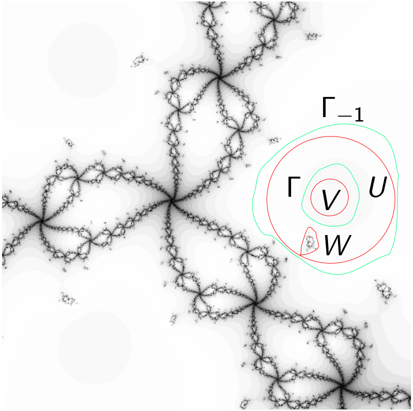

Figure: $A^*_{n,\alpha}(1)$ for $n = 3$ and $\alpha = 0.2 + 1.592i$

Sketch of the proof of Theorem A

These are the main steps of the proof of Theorem A:

(i) Use Γ and Γ^{-1} to construct $f : A^*_{n,\alpha}(1) \to A^*_{n,\alpha}(1)$ quasiregular, and extend it by symmetry to $\hat{\mathbb{C}}$.

(ii) Construct μ both $F-$invariant and $I_\xi-$invariant.

(iii) Normalize the quasiconformal map obtained previously and determine properties of the map, say R, conjugated to $O_{n,\alpha}$.

(iv) Conclude that R is the map obtained by applying Newton's method to f_n.

(v) Compare the Julia sets.

Sketch of the proof of Theorem B for the case $n = 2$

Consider $\alpha > 0$ real. For large enough α, we proceed as follows:

(i) $O_{2,\alpha}$ is conjugated to $z^3 \frac{z-2(\alpha-1)}{1-2(\alpha-1)z} = z^3 \frac{z-a}{1-az}$, for $a = 2(\alpha - 1)$.

(ii) $0 < z_0 < \infty$.

(iii) There exists $p > 0$ such that $[p, \infty) \subset A_a^*(\infty)$.

(iv) Show by computation the existence of a critical point c such that $0 < c < z_0$ and $O_{2,\alpha}(c) > p$.

(v) Prove that c cannot lie in a preimage of $A_a^*(\infty)$, using the Schwarz Reflection Principle and connectivities of Fatou components.

(vi) Use a result proven by Campos-Canela-Vindel to conclude that the parameters in the hyperbolic component containing ∞ behave as described in Theorem A.

Idea of the proof of Theorem B for the case $n \geq 3$

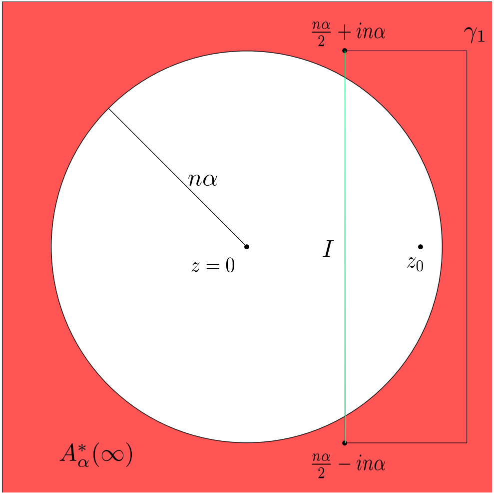

Figure: The zero z_0 is separated by $(I \cup \gamma_1) \subset A^*_{n,\,\alpha}(\infty)$ from $z = 0$.

Thank you for your attention!