Indecomposable Continua in Exponential Dynamics

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Exponential family and continua

We are interested in indecomposable continua appearing in the dynamics of the exponential map E_λ defined by

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Definition

- Continuum is a nonempty compact connected space.
- An indecomposable continuum is a continuum that cannot be represented as the union of two proper subcontinua (two proper subsets that are continuum).

Partition of the complex plane

Definition

Let $(R_j)_{j\in\mathbb{Z}}$ be the partition of the plane defined by

$$R_j = \{z \in \mathbb{C} : (2j-1)\pi - \arg\lambda < \mathrm{Im}(z) \leq (2j+1)\pi - \arg\lambda\}.$$

Let $S : \mathbb{C} \longrightarrow \mathbb{Z}^{\mathbb{N}}$ be a function such that

$$S(z) = (s_0 s_1 s_2 \dots) \iff \forall_{k\in\mathbb{N}} \quad E_\lambda^k(z) \in R_{s_k}.$$

If $S(z) = s$ we call s *itinerary* of z.

Points with the same itinerary

Definition

Let

- $I(s) = \{z \in \mathbb{C} : S(z) = s\}$,
- $\gamma(s) = \{z \in \mathbb{C} : S(z) = s \text{ and } E_\lambda^n(z) \xrightarrow{n\to\infty} \infty\}$.

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Theorem

The set $I(s)$ is nonempty if and only if s is exponentially bounded. Moreover for every exponentially bounded itinerary s the set $\gamma(s)$ contains injective curve $\omega_s : [\zeta, \infty) \longrightarrow \mathbb{C}$ such that $\mathrm{Re}(\omega_s(t)) \xrightarrow{t \to +\infty} +\infty$. We call that curve tail of hair (ray).

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This theorem was investigated by, among others,

- R. Devaney, M. Krych (1984)
- C. Bodelón, R. Devaney, M. Hayes, G. Roberts, L. Goldberg, J. Hubbard (1999)
- D. Schleicher, J. Zimmer (2003)

Existence of indecomposable continua

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Two of them are as follows.

- If $\lambda > \frac{1}{e}$ and s is bounded itinerary with infinitely many blocks of zeros (that are finite but sufficient long) then $I(s) \cup \{\infty\}$ is indecomposable continuum in Riemann sphere.
- If $\lambda > \frac{1}{e}$ and $s = (000\ldots)$ then there is compactification of $I(s)$ that makes it pair of indecoposable continua.

Julia set of exponential map

Theorem (M. Misiurewicz; 1981)

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Remark

For every $\lambda > \frac{1}{e}$ asymptotic value 0 goes to ∞.

Indecomposable continua in the first case

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The main result is as follows.

Let t_m be a block of digits of length m with nonzero first digit and such that each digit has absolute value at most $M \in \mathbb{Z}_+$.
Let 0_m be a block of length m consisting of all zeroes.

Theorem (R. Devaney, X. Jarque)

Let $\lambda > \frac{1}{e}$. Given an infinite sequence of blocks $t_{m_1}, t_{m_2}, \ldots$ (as above) there exist a sequence of integers n_j such that the set $I(s) \cup \{\infty\}$ for the sequence

$$s = t_{m_1}\ 0_{n_1}\ t_{m_2}\ 0_{n_2}\ t_{m_3}\ \ldots\ ,$$

is an indecomposable continuum in the Riemann sphere.

Indecomposable continua in the first case

In order to proof mentioned theorem we are interested in itineraries $s = (s_0 s_1 s_2 \dots)$ such that:

- there exists $M \in \mathbb{Z}_+$ such that $\forall_{k \in \mathbb{N}} |s_k| \leqslant M$,
- s does not end in all zeros i.e $\forall_{K \in \mathbb{N}} \exists_{k \geqslant K} s_k \neq 0$.

We denote the set of such s by Σ_M.

Indecomposable continua in the first case

One of the things that the first condition gives us is that there exists ζ such that for every $s \in \Sigma_M$ the set ω_s is graph over $[\zeta, \infty)$.

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Second condition allow us to construct hair $\gamma(s)$ by

$$\gamma(s) = \bigcup_{n=1}^{\infty} L_{\lambda,s_0} \circ L_{\lambda,s_1} \circ \ldots \circ L_{\lambda,s_{n-1}} \left(\omega_{\sigma^n(s)}\right),$$

where σ is a shift map: $\sigma((s_0 s_1 s_2 \ldots)) = (s_1 s_2 s_3 \ldots)$, and $L_{\lambda,j}$ is branch of inverse of E_λ with values in R_j.

The proof is based on Curry's theorem.

Theorem (S. Curry; 1991)

Suppose X is a one-dimensional plane continuum which is the closure of a ray that limits on itself; and suppose that X separates the plane into not more than finitely many components. Then X is indecomposable.

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In order to use that theorem we want to show that:

- $I(s) \cup \{\infty\}$ a one-dimensional continuum in the Riemann sphere,
- $I(s)$ is the closure of $\gamma(s)$,
- $\mathbb{C} \setminus I(s)$ is connected set,
- $\gamma(s)$ is a curve that limits on itself.

Indecomposable continua in the first case

In order to show the first three properties, we need to analyze the topological properties of the set $I(s)$.

The main theorem of this part is the following characterization of the dynamics of set $I(s)$.

Theorem (R. Devaney, X. Jarque)

Let $s \in \Sigma_M$. Then there is a unique point $z_s \in I(s)$ whose orbit is bounded. All other points have ω-limit sets that are either the point at ∞ or the orbit of 0 together with ∞.

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Let's focus on the last property.

Definition

The curve $\varphi : [0, +\infty) \longrightarrow \hat{\mathbb{C}}$ limits on itself if for every $t \in [0, +\infty)$ there exists a sequence $(t_j)_{j=0}^{\infty}$ such that $t_j \xrightarrow{j \to \infty} +\infty$ and $\varphi(t_j) \xrightarrow{j \to \infty} \varphi(t)$.

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We want to show that, for some s, $\gamma(s)$ limits on itself.

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$$V(a,b) = \{z \in \mathbb{C} : \mathrm{Re}(z) \in [a-1, b+1],\ |\,\mathrm{Im}(z)| \leqslant (2M+1)\pi\}.$$

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We can find increasing sequences a_n and b_n such that for every n and k

$$V(a_{n+1}, b_{n+k+1}) \subseteq E_\lambda(V(a_n, b_{n+k}))$$

and for every $s \in \Sigma_M$ rectangle $V(a_n, b_{n+k})$ contains $E_\lambda^n(\alpha_s^k)$, where α_s^k is the initial part of the tail ω_s (the greater k the longer that initial part α_s^k and $\bigcup_{k=0}^{\infty} \alpha_s^k = \omega_s$).

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We have the following inclusions:

$$\alpha_s^k \subseteq \bigcap_{n=1}^{\infty} L_{\lambda,s_0} \circ L_{\lambda,s_1} \circ \ldots \circ L_{\lambda,s_{n-1}}\left(V(a_n, b_{n+k})\right) \subseteq \omega_s.$$

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Now we apply the function $L_{\lambda,0}$ to the set $\omega_{0_l t}$ as many times as necessary and, for any n and j, we obtain the curve that is part of $\gamma(0_k t)$ and that passes twice through $V(a_n, b_{n+j})$.

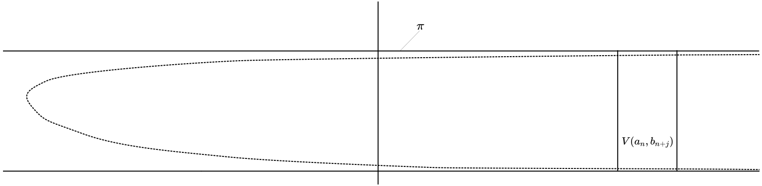

Figure: Part of the hair $\gamma(0_k t)$ that passes twice through $V(a_n, b_{n+j})$.

Indecomposable continua in the first case

It this case for any finite, and bounded by M, sequence $s_0, s_1, \ldots, s_{n-1}$ two parts of the hair $\gamma(s_0 s_1 \ldots s_{n-1} 0_k t)$ are in

$$L_{\lambda,s_0} \circ L_{\lambda,s_1} \circ \ldots \circ L_{\lambda,s_{n-1}} \left(V(a_n, b_{n+j})\right),$$

which makes them close to each other (the larger n, the smaller this distance). One of those parts is the initial part α_s^k.

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Now for every j we divide itinerary s into three parts:

$$s = \underbrace{t_{m_1}\, 0_{n_1} \ldots t_{m_{j-1}}}_{s_0 s_1 \ldots s_{q_j - 1}} 0_{n_j} \underbrace{t_{m_{j+1}}\, 0_{n_{j+1}} \ldots}_{t},$$

and we set the length n_j of the block 0_{n_j} so that the hair $\omega(0_{n_j} t)$ passes twice through a long enough rectangle which is far enough to the right (both length and distance increase with j).

In this way we get that $\gamma(s)$ limits on tail $\omega(s)$.

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Analogous reasoning for $\sigma^n(s)$ gives us that $\gamma(s)$ limits on itself.

Work in progress

Construction of indecomposable continua for unbounded itineraries.

Hausdorff dimension

Theorem (Ł. Pawelec, A. Zdunik; 2014)

The Hausdorff dimension of indecomposable continuum just constructed is equal to 1.

Indecomposable continua in the second case

The case of $\lambda > \frac{1}{e}$ and $s = (000\ldots)$ was described by R. Devaney (1999).

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If $S(z) = (000\ldots)$ and $\mathrm{Im}(z) \geqslant 0$ then $\mathrm{Im}(E_\lambda^n(z)) \geqslant 0$ for all $n \in \mathbb{N}$.

Thus we are interested in set

$$\Lambda = \{z \in \mathbb{C} : \forall_{n\in\mathbb{N}}\ E_\lambda^n(z) \in S\},$$

where $S = \{z \in \mathbb{C} : 0 \leqslant \mathrm{Im}(z) \leqslant \pi\}$.

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Let

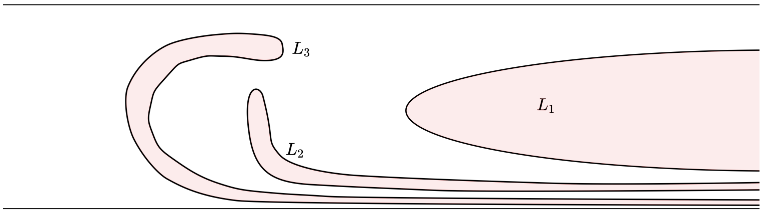
$$L_n = \{z \in S : \forall_{0\leqslant j\leqslant n}\, E_\lambda^j(z) \in S \text{ and } E_\lambda^{n+1}(z) \notin S\}.$$

We have

$$\Lambda = S \setminus \bigcup_{n=1}^{\infty} L_n.$$

Indecomposable continua in the second case

Let B_n be a boundary of L_n. Every set B_n is a curve such that $E_\lambda^n(B_n) = \{z \in \mathbb{C} : \mathrm{Im}(z) = \pi\}$.



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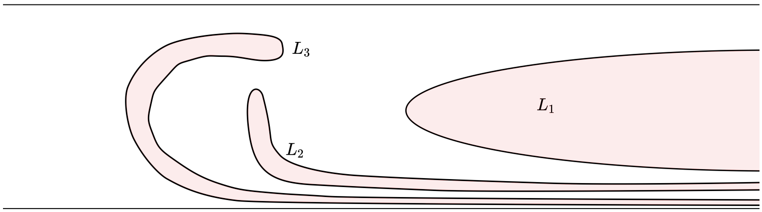

Lemma

For every $n > 0$ set $\bigcup_{j=n}^{\infty} B_j$ is dense in Λ.

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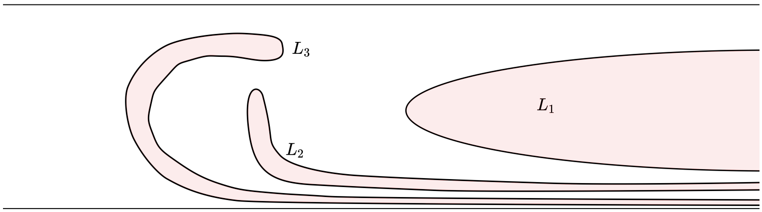

Lemma

For every $n > 0$ set $\bigcup_{j=n}^{\infty} B_j$ is dense in Λ.

In this case we want to use Curry's theorem too.

We create compactification of Λ by gluing the ends of the B_n curves.

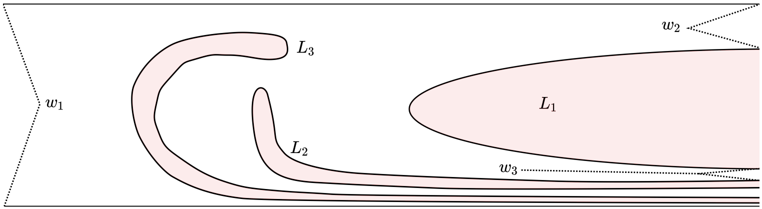

Figure: Idea of creating Γ which is compactification of Λ.

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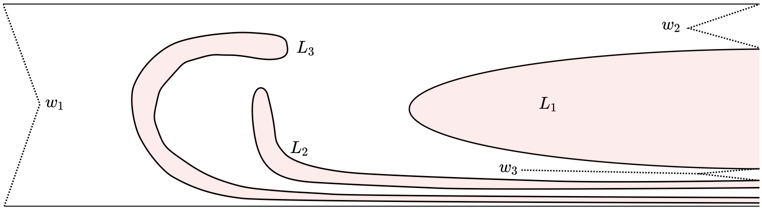

Figure: Idea of creating Γ which is compactification of Λ.

Previous lemma gives us that new curve limits on itself and compactification Γ of Λ is indecomposable continuum.

Hausdorff dimension

Theorem (Ł. Pawelec, A. Zdunik; 2014)

The Hausdorff dimension of Λ is equal to 1.

Indecomposable continua in other cases

Indecomposable continua appear also in other cases.

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If $\lambda = 2\pi i$ then 0 is preperiodic: $E_\lambda(0) = 2\pi i$, $E_\lambda(2\pi i) = 2\pi i$. Thus $\lambda = 2\pi i$ is Misiurewicz paramter. For such λ we have $J(E_\lambda) = \mathbb{C}$.

Theorem (R. Devaney, X. Jarque, M. Rocha; 2005)

If $\lambda = 2\pi i$ then there exist itineraries s such that the set $I(s)$ consists of either an indecomposable continuum in the Riemann sphere and a distinguished curve that accumulates on it, or else the closure of a pair of curves that accumulate on themselves as well as on each other. In this case the set of accumulation points is an indecomposable continuum.

There are some general results.

Theorem (L. Rempe; 2007)

Suppose that E_κ ($E_\kappa(z) = e^z + \kappa$) is an exponential map whose singular value κ is on a dynamic ray or is the landing point of such a ray. Then there exist uncountably many dynamic rays g whose accumulation set (on the Riemann sphere) is an indecomposable continuum containing g.

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Indecomposable continua exist for functions other than E_λ. For $f(z) = z + e^{-z}$ we have the following result.

Theorem (N. Fagella, A. Jové; 2023)

There exist uncountably many dynamic rays which do not land. The landing set of the non-landing rays is an indecomposable continuum.

Thank you!