Periodic boundary points for transcendental maps

(joint work with N. Fagella)

1. Introduction

$f: \mathbb{C} \to \hat{\mathbb{C}}$ meromorphic function

$\mathcal{F}(f)$: Fatou set; $\mathcal{J}(f)$: Julia set

<u>THM</u> (Fatou, Julia) Periodic points are dense in $\mathcal{J}(f)$

<u>Q</u> Given U invariant Fatou component, are periodic points dense in ∂U?

→ Why this is not obvious?

U Siegel disk ($f|_U$ irrational rotation)
∂U is a Jordan curve } ⇒ no periodic points in ∂U

2. INVARIANT Fatou components

polynomials	transcendental entire
· simply connected · Siegel disks attr / parab basins	+ Baker domains (i.e. $f|_U \rightrightarrows \infty$, ess. sing)
rational	**transcendental meromorphic**
· 1, 2, or ∞ - connected + Hermann rings	

3. – Periodic boundary points for rational maps (Przytycki – Zdunik)

· U Siegel disk / Hermann ring → don't expect periodic points in ∂U
(if ∂U locally connected, $f|_{\partial U} \sim$ irrational rotation ⇒ no periodic points)

· U attr / parab basin → we expect periodic points in ∂U

U simply connected, ∂U locally connected

$\Rightarrow f|_{\partial U} \sim (\theta \mapsto d\theta \mod 1) \Rightarrow$ periodic points dense in ∂U

<u>THM</u> (Przytycki - Zdunik) f rational, U attr / parab basin $\Rightarrow$ periodic points dense in ∂U

↳ two different proofs
- $\boxed{\text{simply connected attr basins}}$
- general case (geometric coding trees)

4. Generalization to transcendental maps (simply connected FC)

(Fagella - J)

<u>Q</u> 1. Which FC? attr / parab basins + Baker domains?

2. How to adapt the proof of PZ to the transcendental setting?

<u>THM</u> (FJ) f meromorphic, U $\begin{cases} \text{attr / parab basin} \\ \text{doubly parab BD} \end{cases}$ simply connected

If (a): $\omega_U(P(f) \cap \partial U) = 0$

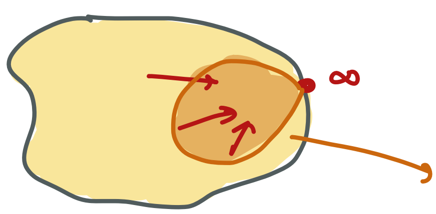
(b): $\overline{SV(f) \cap U} \subseteq U$

→ periodic points dense in ∂U

<u>Obs</u>: (a), (b) always hold for rational maps $\begin{cases} \text{(a) Pesin's theory} \\ \text{(b) } \# SV(f) < \infty \end{cases}$

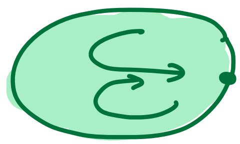
<u>Q1</u> U Baker domain, do we expect periodic points in ∂U? Depend

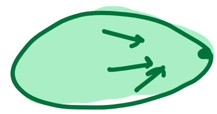
Classification of BD

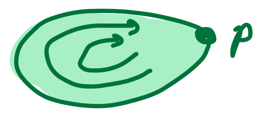


→ NO normal form around ∞ !! (essential sing)

there exists a "petal" → three possible dynamics doubly parabolic | hyperbolic | simply parabolic







(~ parabolic petal) | (as if it was attracting) | (half of a parab petal)

↓

like a parabolic basin
(if $\overline{SV(f)\cap U} \subset U$, same ergodic properties)

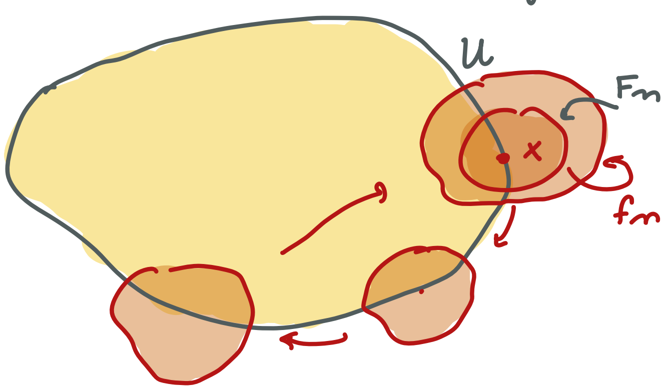
Q2 Proof of P2 (rational, simply connected) → generalization to transcendental

1. · For ω_U - ae $x \in \partial U$, $\exists\, r := r(x) > 0$ st. all inverse branches well-def in $D(x, r)$ + contracting
(wrt some metric)

rational maps → Pesin's theory + Euclidean metric

trans. maps → $\omega_U(P(f) \cap \partial U) = 0$ + hyperbolic metric for $\hat{\mathbb{C}} - P(f)$

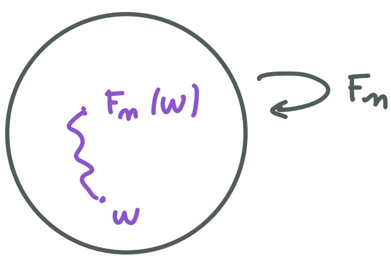
· ae $x \in \partial U$ is recurrent
(general fact for the FC we consider)



F_n : inverse branch

⇒ periodic point p in $D(x, r)$

2. How to ensure $p \in \partial U$?

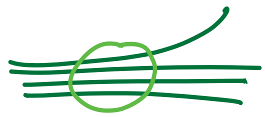


* We need to find $w \in D \cap U$ st. $F_n(w) \in D \cap U$
and connect w and $F_n(w)$ with a curve γ in $U \cap D$

→ If we can do this, $\bigcup_{m \geq 0} F_n^m(\gamma) \subseteq U$ lands at the periodic point p ⇒ $p \in \partial U$

*1) <u>Not obvious</u> $D \cap U$ may not be connected

<u>Ex.</u> ∂U Cantor bouquet



<u>TOOLS</u> :

$$\begin{tikzcd} U \arrow[r, "f"] & U \\ \mathbb{D} \arrow[u, "\varphi"] \arrow[r, "g"] & \mathbb{D} \arrow[u, "\varphi"'] \end{tikzcd}$$

φ → Riemann map

g ↑ associated inner function

→ extension to the boundary

$$\begin{tikzcd} \partial U \arrow[r, "f"] & \partial U \\ \partial \mathbb{D} \arrow[u, "\varphi^*"] \arrow[r, "g^*"'] & \partial \mathbb{D} \arrow[u, "\varphi^*"'] \end{tikzcd}$$

in the sense of radial limits almost everywhere

\+

Koebe distortion

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measure - theoretical properties

... How to extend to multiply connected? understanding π