Peschl–Minda Derivatives

on the $\mathbb{D}$isk, the $\mathbb{S}$phere and Beyond

Michael Heins

Julius–Maximilians–Universität Würzburg

June 22, 2023

joint work with A. Moucha, O. Roth & T. Sugawa

Peschl–Minda derivatives on $\mathbb{D}$

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- Explicit formula **(Aharonov 1969)**:

$$D^n f(z) = \sum_{k=1}^{n} \frac{n!}{k!} \binom{n-1}{k-1} (-\overline{z})^{n-k} \left(1-|z|^2\right)^k f^{(k)}(z) . \quad (1)$$

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- Replace $'$ by ∂ in (1) **(Kim & Sugawa 2011)**.

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$$D^n f(z) = \partial^n (f \circ \Phi_z)\big|_0$$

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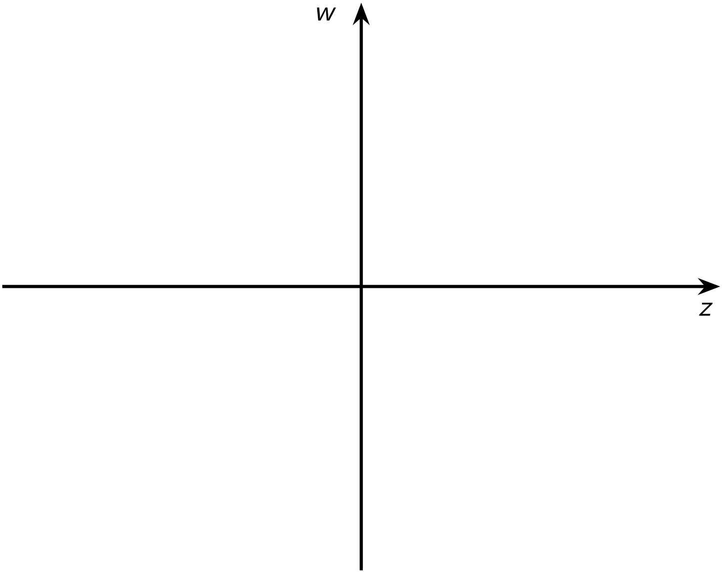
with

$$\Omega := \left\{(z,w) \in \widehat{\mathbb{C}}^2 : z \cdot w \neq 1\right\}.$$

Peschl–Minda derivatives on Ω

A schematic picture of Ω



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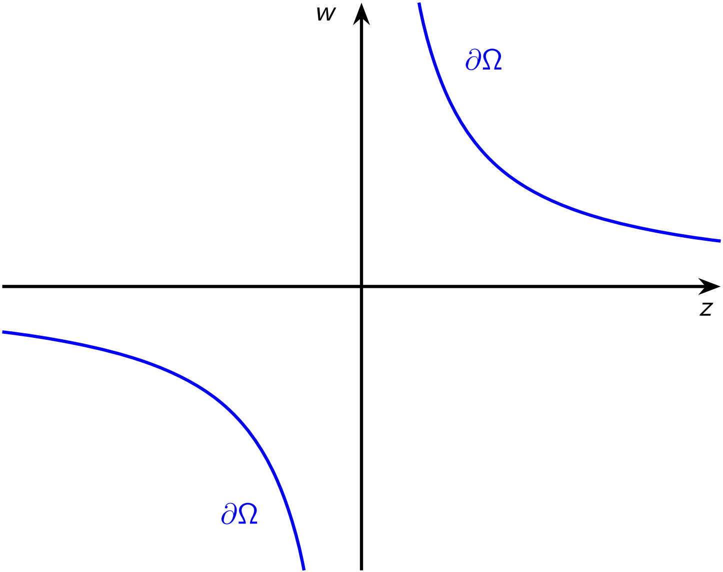

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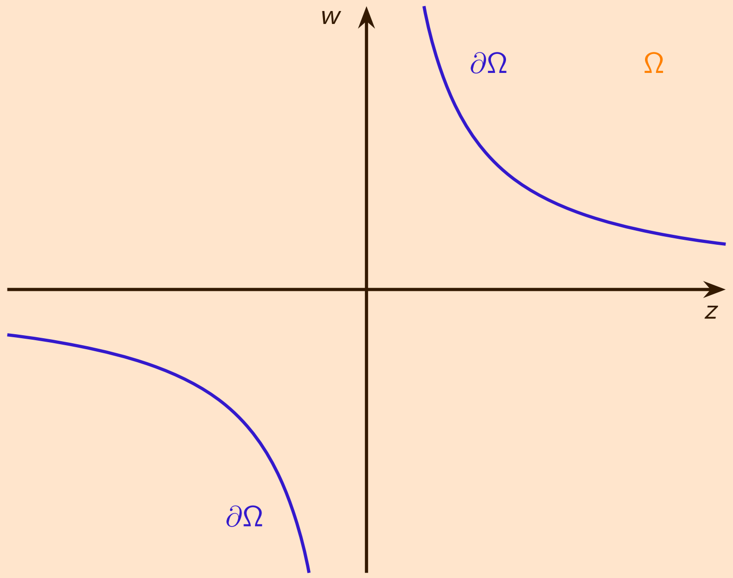

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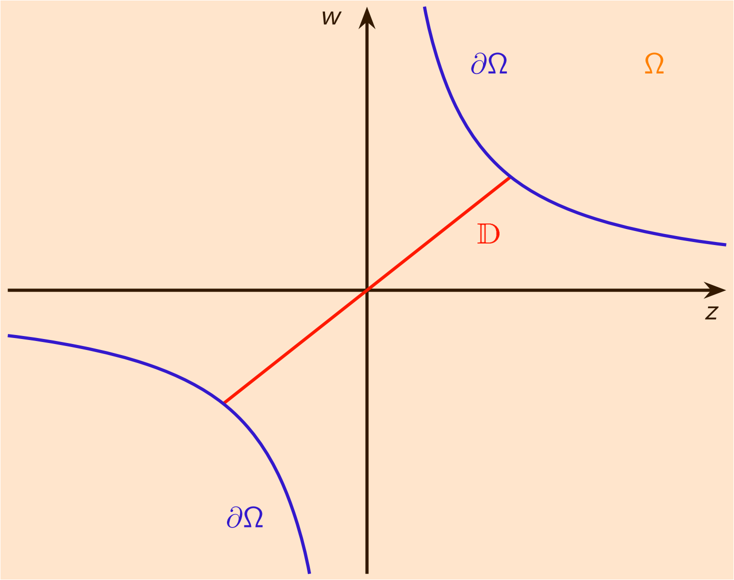

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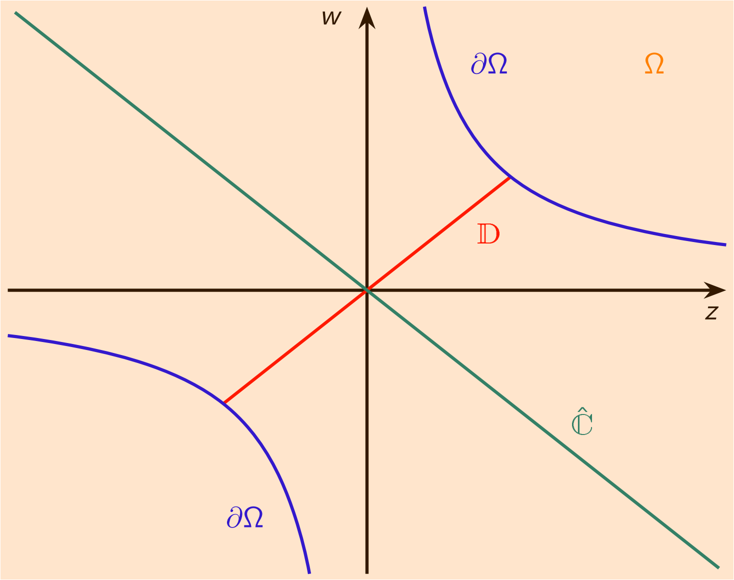

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Theorem (Moucha, Roth, Sugawa, H.)

Recursion identity

$$D^{m+1,n} = w(n-m)D^{m,n} + D^{1,0} \circ D^{m,n} + n(n-1)D^{m,n-1}.$$

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If $m \geq n$, then there exist $a_k(m,n) \in \mathbb{N}_0$ s.t.

$$D^{m,n} = D^{m-n,0} \circ \left(\left(D^{1,1}\right)^n + \sum_{k=1}^{n-1} a_k(m,n) \left(D^{1,1}\right)^k \right)$$

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$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\hbar^n}{(1-\hbar)\cdots(1-(n-1)\hbar)} (D^{0,n} f) \cdot (D^{n,0} g)$$

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converges absolutely in $\mathscr{H}(\mathscr{D} \times \Omega)$. For every $\hbar \in \mathscr{D}$, the triple $(\mathscr{H}(\Omega), +, \star_\hbar)$ is a Fréchet algebra with respect to the topology of locally uniform convergence on Ω.

Last slide.

This is the last slide. Take a look at the piece of paper you wrote your questions and suggestions on. It is probably empty. On the off–chance that it is not, the time has come to make yourself heard!