Iteration in tracts

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Outline

- The escaping set.
- Rates of escape and tracts.
- Slow escape within a logarithmic tract.
- Slow escape in more general tracts.

The escaping set

Definition

Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function, then the **escaping set** $I(f)$ is

$$I(f) = \{z : f^n(z) \to \infty \text{ as } n \to \infty\}.$$

- Eremenko (1989) showed $I(f)$ has the following properties:
  - $J(f) = \partial I(f)$
  - $I(f) \cap J(f) \neq \emptyset$,
  - $\overline{I(f)}$ has no bounded components.
- Eremenko's conjecture: All components of $I(f)$ are unbounded.

Fast escape

- First introduced by Bergweiler and Hinkkanen (1999)

Definition

The **fast escaping set**,

$$A(f) = \{z : \text{ there exists } L \in \mathbb{N} \text{ such that } |f^{n+L}(z)| \geq M^n(R) \text{ for } n \in \mathbb{N}\}$$

where

$$M(R) = \max_{|z|=R} |f(z)| \quad \text{for } R > 0.$$

- $\partial A(f) = J(f)$
- $A(f) \cap J(f) \neq \emptyset$
- All components of $A(f)$ are unbounded by a result of Rippon and Stallard (2005).

Slow escape

- There exist points that escape arbitrarily slowly.

Theorem (Rippon, Stallard, 2011)

Let f be a transcendental entire function. Then, given any positive sequence (a_n) such that $a_n \to \infty$ as $n \to \infty$, there exist

$$\zeta \in I(f) \cap J(f) \text{ and } N \in \mathbb{N}$$

such that

$$|f^n(\zeta)| \leq a_n, \quad \text{for } n \geq N.$$

- $A(f)$ is always different from $I(f)$.

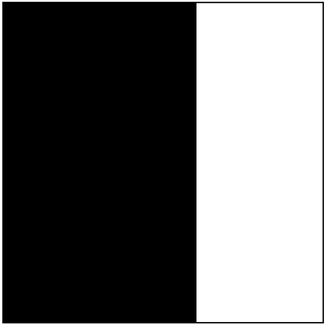
Tracts

Definition

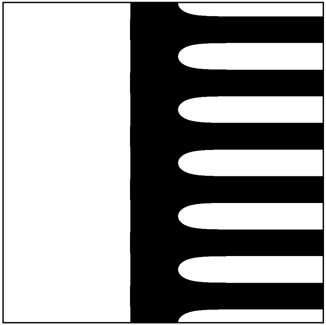
Let D be an unbounded domain in $\mathbb{C}$ whose boundary consists of piecewise smooth curves. Further suppose that the complement of D is unbounded and let f be a complex valued function whose domain of definition includes the closure $\bar{D}$ of D.
Then, D is a **direct tract** if f is analytic in D, continuous on $\bar{D}$, and if there exists $R > 0$ such that $|f(z)| = R$ for $z \in \partial D$ while $|f(z)| > R$ for $z \in D$. If in addition the restriction $f : D \to \{z \in \mathbb{C} : |z| > R\}$ is a universal covering, then D is a **logarithmic tract**.

- Every transcendental entire function has a direct tract.

Examples

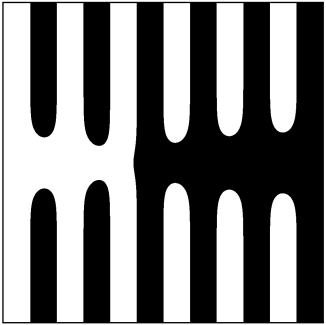


$\exp(z)$

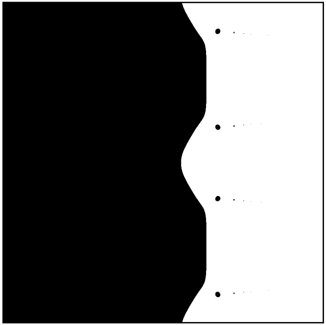


$\exp(\exp(z) - z)$

More examples



$\exp(\sin(z) - z)$



$\exp(\exp(z)) - \exp(z)$

Logarithmic transform and the expansion estimate

Let D be a logarithmic tract, f holomorphic in D, and suppose that $f(D) = \mathbb{C} \setminus \overline{\mathbb{D}}$ with $f(0) \in \mathbb{D}$. We consider the logarithmic transform of f defined by the following commutative diagram,

$$\begin{tikzcd} \log D \arrow[r, "F"] \arrow[d, "\exp"'] & H \arrow[d, "\exp"] \\ z \arrow[r, "f"] & w \end{tikzcd}$$

where $\exp(F(t)) = f(\exp(t))$ for $t \in \log D$ and $H = \{z : \mathrm{Re}(z) > 0\}$.

Lemma (Eremenko, Lyubich 1992)

For $z \in D$ as above, we have

$$\left| \frac{z f'(z)}{f(z)} \right| \geq \frac{1}{4\pi} \log |f(z)| \,.$$

Slow escape in logarithmic tracts

Lemma

For a logarithmic tract D and r sufficiently large so that $M_D(r) > e^{16\pi^2}$,

$$f(A(r, 2r) \cap D) \supset \bar{A}(e^{16\pi^2}, M_D(r)).$$

Theorem

Let f be a transcendental entire function with a logarithmic tract D. Then, given any positive sequence (a_n) such that $a_n \to \infty$ as $n \to \infty$, there exist

$$\zeta \in I(f) \cap J(f) \cap \overline{D} \text{ and } N \in \mathbb{N}$$

such that

$$f^n(\zeta) \in \overline{D}, \text{ for } n \geq 1,$$

and

$$|f^n(\zeta)| \leq a_n, \text{ for } n \geq N.$$

Two-sided slow escape in logarithmic tracts

Theorem

Let f be a transcendental entire function with a logarithmic tract D. Then, given any positive sequence (a_n) such that $a_n \to \infty$ as $n \to \infty$ and $a_{n+1} = O(M_D(a_n))$ as $n \to \infty$, for any $C > 1$, there exist

$$\zeta \in J(f) \cap \overline{D}, \; and \; N \in \mathbb{N},$$

such that

$$f^n(\zeta) \in \overline{D}, \; \textit{for } n \geq 1,$$

and

$$a_n \leq |f^n(\zeta)| \leq C a_n, \; \textit{for } n \geq N.$$

Hyperbolic distance

Definition

Let $\mathbb{D}$ be the unit disc. The hyperbolic distance on $\mathbb{D}$ is

$$\rho_{\mathbb{D}}(z_1, z_2) = \inf_{\gamma} \int_{z_1}^{z_2} \frac{|dz|}{1 - |z|^2}$$

where this infimum is taken over all smooth curves γ joining z_1 to z_2 in $\mathbb{D}$.

Annulus covering using hyperbolic contraction

Lemma

Let Σ *be a hyperbolic Riemann surface. For a given* $K > 1$*, if* $f : \Sigma \to \mathbb{C} \setminus \{0\}$ *is analytic, then for all* $z_1, z_2 \in \Sigma$ *such that*

$$\rho_\Sigma(z_1, z_2) < \frac{1}{2} \log\left(1 + \frac{\log K}{10\pi}\right) \quad \textit{and} \quad |f(z_2)| \geq K|f(z_1)|$$

we have

$$f(\Sigma) \supset \bar{A}(|f(z_1)|, |f(z_2)|).$$

Slow escape in 'nice' direct tracts

Theorem

Let f be a transcendental entire function and let D be a direct tract of f, bounded by 'nice' curves. Then, given any positive sequence (a_n) such that $a_n \to \infty$ as $n \to \infty$, there exist

$$\zeta \in I(f) \cap J(f) \cap \overline{D} \text{ and } N \in \mathbb{N}$$

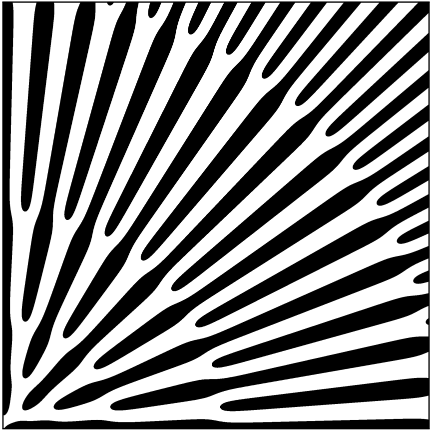
such that

$$f^n(\zeta) \in \overline{D}, \text{ for } n \geq 1,$$

and

$$|f^n(\zeta)| \leq a_n, \text{ for } n \geq N.$$

Example of slow escape in a non-logarithmic tract



$$\exp\left(-\sum_{k=1}^{\infty}\left(\frac{z}{2^k}\right)^{2^k}\right)$$

Thank you for your attention!