Automorphisms of $\mathbb{C}^2$ with multiply connected attracting cycles of Fatou components

Josias Reppekus (Università degli Studi di Roma "Tor Vergata")

Topics in Complex Dynamics 2019 - From combinatorics to transcendental dynamics, Barcelona, March 25-29, 2019

26. March 2019

Fatou components

Definition

Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be a holomorphic map. The *Fatou set* of F is the open set

$$\mathcal{F} := \{z \in \mathbb{C}^d \mid \{F^n\}_{n\in\mathbb{N}} \text{ is normal in a neighbourhood of } z\}.$$

A *Fatou component* of F is a connected component U of $\mathcal{F}$ and it is

- *invariant* if $F(U) = U$,
- *attracting* if $(F|_U)^n \to p \in \overline{U}$ (in particular $F(p) = p$),
- *non-recurrent* if no orbit starting in U accumulates in U (in the attracting case, i.e. $p \in \partial U$).

Main result

Theorem (Bracci-Raissy-Stensønes)

There exists $F \in \mathrm{Aut}(\mathbb{C}^d)$ *with an invariant, attracting, non-recurrent Fatou component biholomorphic to* $\mathbb{C} \times (\mathbb{C}^*)^{d-1}$ *attracted to the origin* O.

Theorem (R)

Let $k, p \in \mathbb{N}^*$. *There exists* $F \in \mathrm{Aut}(\mathbb{C}^d)$ *with k disjoint, p-periodic cycles of attracting, non-recurrent Fatou components biholomorphic to* $\mathbb{C} \times (\mathbb{C}^*)^{d-1}$ *attracted to the origin* O.

Other results

Theorem (R)

Let $k, p \in \mathbb{N}^*$. There exists $F \in \mathrm{Aut}(\mathbb{C}^d)$ with k disjoint, p-periodic cycles of attracting, non-recurrent Fatou components biholomorphic to $\mathbb{C} \times (\mathbb{C}^*)^{d-1}$ attracted to the origin O.

Other results

Let $U \subseteq \mathbb{C}^2$ be an invariant attracting Fatou component of $F \in \mathrm{Aut}(\mathbb{C}^2)$.

1. If U is recurrent or F is polynomial, then $U \sim \mathbb{C}^2$ (Rosay-Rudin/Peters-Vivas-Wold, Ueda/Peters-Lyubich).
2. U is Runge in $\mathbb{C}^2$ (Ueda).
   In particular, $H^2(U) = 0$ (Serre).

A one-resonant morphism

Take $F : \mathbb{C}^2 \to \mathbb{C}^2$ of the form

$$F(z,w) = \begin{pmatrix} \lambda z \\ \overline{\lambda} w \end{pmatrix} \left(1 - \frac{zw}{2}\right) + O(\|(z,w)^l\|),$$

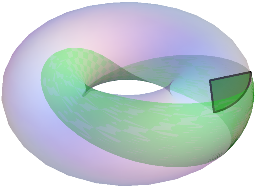
where $|\lambda| = 1$ is not a root of unity and λ is Brjuno (in particular $\lambda\overline{\lambda} = 1$).

This is a so-called *one-resonant* map (Bracci-Zaitsev). F acts on the one-dimensional coordinate $u = zw$ as

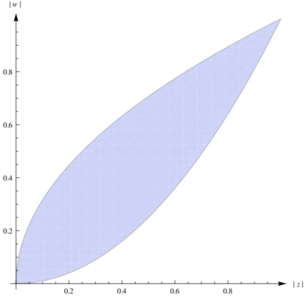
$$u \mapsto u(1 - u + 1/4u^2) + O(\|(z,w)^l\|).$$

Local Dynamics

(z, w)



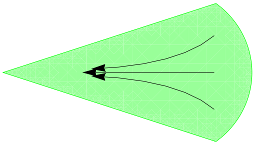
$\times$



$(\arg z, \arg w)$

$\|(z,w)\| < |u|^\beta$

$u = zw$



$=: S$

$$u \mapsto u(1 - u + 1/4u^2) + O(\|(z,w)^l\|)$$

Theorem (Bracci-Zaitsev 2013)

A Fatou component

Let $\Omega := \bigcup_{n\in\mathbb{N}} F^{-n}(B)$. Then Ω is completely F-invariant, open and attracted to O.

Claim

Ω is a (union of) Fatou component(s).

Proof.

Ingredients:

1. For $(z, w) \in \Omega$, we have $|z_n| \sim |w_n|$.
2. If λ is Brjuno, then there exist local coordinates tangent to (z, w) such that $(\mathbb{D}, 0)$ and $(0, \mathbb{D})$ are *Siegel discs*, that is, analytic discs on which F acts as an irrational rotation (Pöschel). $\square$

Globalisation

Theorem (Forstnerič)

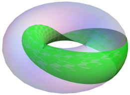
There exists an automorphism $F \in \mathrm{Aut}(\mathbb{C}^2)$ *of the form*

$$F(z,w) = \begin{pmatrix} \lambda z \\ \overline{\lambda} w \end{pmatrix} \left(1 - \frac{zw}{2}\right) + O(\|(z,w)^l\|).$$

Claim

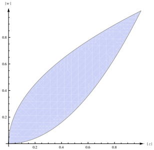
For F as above, $\Omega \cong \mathbb{C} \times \mathbb{C}^*$.

Local Fatou coordinates on B



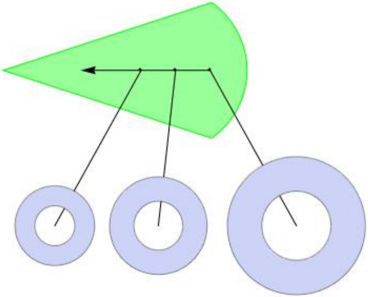
(arg z, arg w)

$\times$



$\|(z, w)\| < |u|^{\beta}$

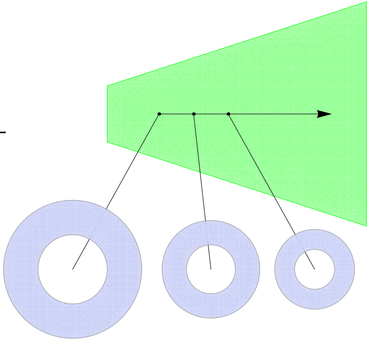
$u = zw$



w

$U = 1/u$

$U \mapsto U + 1 + HOT$



w

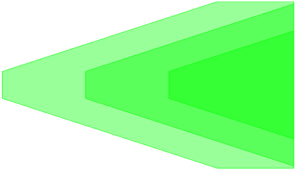
Fatou coordinates:

$$\tilde{U} \mapsto \tilde{U} + 1$$

$$\tilde{w} \mapsto \overline{\lambda} e^{-\frac{1}{2\tilde{U}}}$$

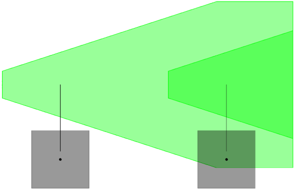
Extending Fatou coordinates

$\tilde{U}$ extends via $\tilde{U} \mapsto \tilde{U} + 1$ to $\mathbb{C}$:

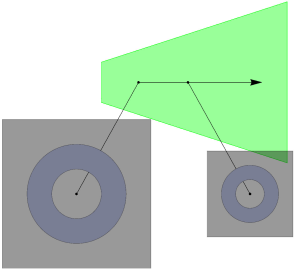


$\tilde{w}^{(n)}(p) = \tilde{w}(F^n(p))$ is defined over $H_n = H - n$.

$\rightsquigarrow$ $\mathbb{C}^*$-bundle structure over $\mathbb{C}$:



$\tilde{w}$ extends via $\tilde{w} \mapsto \overline{\lambda} e^{-\frac{1}{2\tilde{U}}}$ to $\mathbb{C}^*$ over $H = \tilde{U}(B)$:



Transition functions:

$$\tilde{w}^{(n+1)} = \lambda e^{\frac{1}{2(\tilde{U}+n)}} \tilde{w}^{(n)}.$$

Such a bundle is trivial

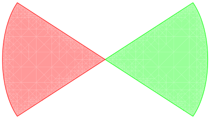
$\rightsquigarrow \Omega \cong \mathbb{C} \times \mathbb{C}^*$

Higher orders and periodic components

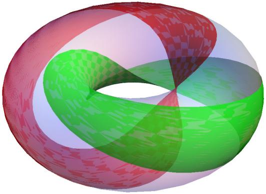
As in the one-dimensional case, we can find multiple such components, invariant or periodic, if we replace F by

$$F(z,w) = \begin{pmatrix} \lambda z \\ \zeta_p \overline{\lambda} w \end{pmatrix} \left(1 - \frac{(zw)^k}{2k}\right) + O(\|(z,w)^l\|),$$

where ζ_p is a p-th root of unity.



$\rightsquigarrow$



Open questions

Classification of Fatou components

Let Ω be a non-recurrent, attracting Fatou component for $F \in \mathrm{Aut}(\mathbb{C}^d)$.

1. Is Ω biholomorphic to $\mathbb{C}^d$, $\mathbb{C}^{d-1} \times \mathbb{C}^*$, ... or $\mathbb{C} \times (\mathbb{C}^*)^{d-1}$?
2. Does the Kobayashi metric vanish: $k_\Omega \equiv 0$?
3. Is there a Fatou coordinate $\psi : \Omega \to \mathbb{C}$ that is also a fibre bundle.

For $d = 2$, 2 and 3 would imply 1.

Extension of Siegel discs

In our example, there are Siegel discs for F tangent to the axes.

1. Can these be extended to entire Siegel curves?
2. Can they be globally simultaneously linearised? I.e. can we find global Pöschel coordinates for F?

Thank you!