On a class of transcendental entire functions

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TCD2019, Barcelona, 25^{th} March, 2019

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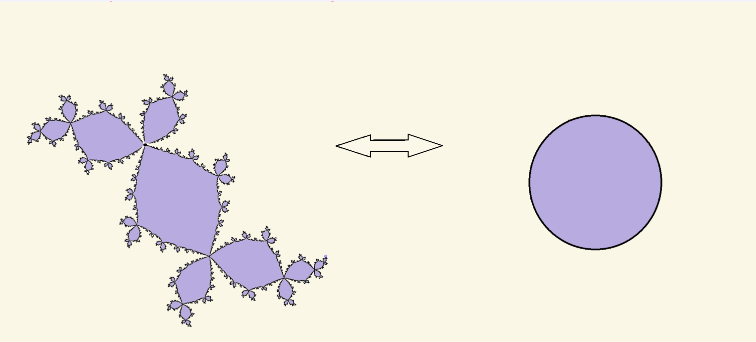

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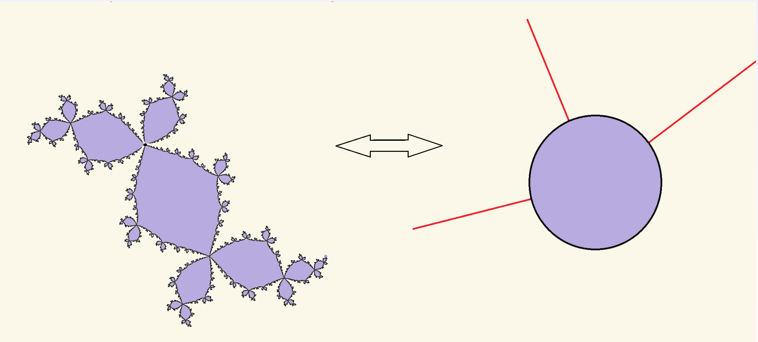

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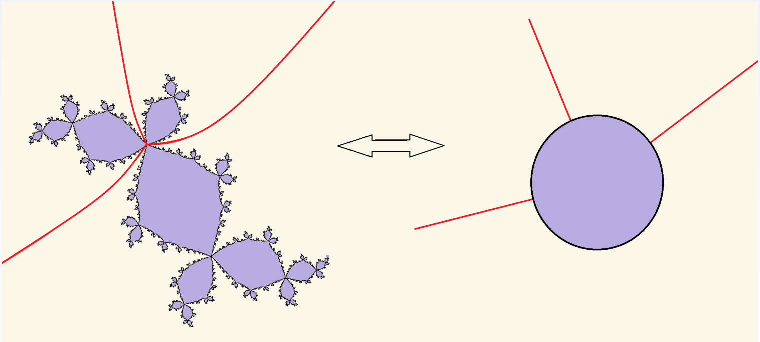

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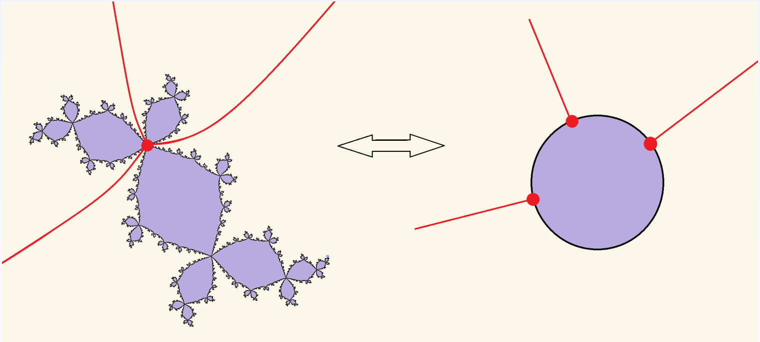

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1. **Dynamic rays** as preimages of radial lines under Böttcher's map.
2. If $J(p_d)$ is connected, then **all rays land** if and only if $J(p_d)$ is locally connected.
3. Dynamics better understood using the *simpler map* z^d. (In particular, *Douady's Pinched Disk* ***model***.)

Transcendental dynamics

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- Not in general; counterexample in ([RRRS] 11')
- Yes for functions of finite order in class $\mathcal{B}$. (Barański 07')([RRRS]).

Singular values

The set of **singular values** $S(f)$ is the smallest closed subset of $\mathbb{C}$ such that $f : \mathbb{C} \setminus f^{-1}(S(f)) \to \mathbb{C} \setminus S(f)$ is a covering map.

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The **postsingular set** of f is defined as

$$P(f) = \overline{\bigcup_{n \geq 0} f^n(S(f))}.$$

- ⋆ $f^k : \mathbb{C} \setminus \mathcal{O}^-(S(f)) \to \mathbb{C} \setminus P(f)$ is a covering map for all $k \geq 0$.

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- We say that γ **lands** at z if $\lim_{t\to 0} \gamma(t) = z$.

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- For certain functions of finite order with bounded postsingular set, all dynamic rays land.
- Not always dynamic rays land. For example, exponential map with escaping singular value. (Rempe-Gillen '07).

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- Model in the *exponential family* for attracting and parabolic orbits. (Rempe-Gillen '06).
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- Model for *strongly subhyperbolic* functions. (Mihaljević-Brandt'12) .

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- Dynamic rays "split" at the critical points $\{\pi i k, k \in \mathbb{Z}\}$.

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3. there exists a *topological model* for the dynamics of the function on its Julia set.

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Theorem B

If f is strongly postcritically separated, then f is *expanding* with respect to some conformal metric that admits a discrete set of *cone singularities*.

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Moreover, the following analogue of Böttcher's Theorem holds:

Theorem (Rempe-Gillen '09)

There exist a constant $R > 0$ and a quasiconformal map $\vartheta : \mathbb{C} \to \mathbb{C}$ such that $\vartheta \circ f = g_\lambda \circ \vartheta$ for all $z \in J_R(g)$, with

$$J_R(g_\lambda) := \{z \in \mathbb{C} : |g_\lambda^n(z)| \geq R \text{ for all } n \geq 1\}.$$

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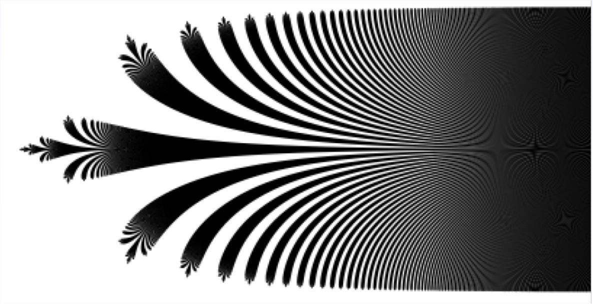

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- If f is of finite order and of disjoint type then $J(f)$ is a *Cantor Bouquet.* [B,J, R-G '12]
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Proposal

To study the class of maps

$$\mathcal{CB} = \left\{ \begin{array}{c} f \in \mathcal{B} : \text{ exists } \lambda \in \mathbb{C} : g_\lambda = \lambda f \text{ is of disjoint type} \\ \text{and } J(g_\lambda) \text{ is a Cantor Bouquet.} \end{array} \right\}$$

Results

Theorem A

Let $f \in \mathcal{CB}$ and strongly postcritically separated. Then

1. every point in their $J(f)$ is either in a dynamic ray or it is the landing point of at least one ray,
2. every dynamic ray of f lands, and
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4. The **endpoints** of X are **dense** in X.
5. If $x \in X$ is accessible from $\mathbb{C} \setminus X$, then x is an endpoint of X. (Equivalently, **every hair** of X is **accumulated** on **by other hairs** from both sides.)

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If in addition g is of disjoint type, then each hair of $J(g)$ is a dynamic ray together with its endpoint.

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3. **For any sequence $y_n \to y$, the arcs $[y_n, \infty)$ converge to $[y, \infty)$ in the Hausdorff metric. ??**
4. **The endpoints of $J(f)$ are dense in $J(f)$. ??**
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Criniferous vs Cantor Bouquet

If f is disjoint type then

$$J(f) \text{ Cantor Bouquet} \Longrightarrow f \text{ criniferous.}$$

Criniferous vs Cantor Bouquet

If f is disjoint type then

$$J(f) \text{ Cantor Bouquet} \implies f \text{ criniferous.}$$

Open question:

$$f \text{ criniferous } \implies J(f) \text{ Cantor Bouquet?}$$

Thanks for your attention!