

Fixed points of post-critically algebraic endomorphisms

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Motivation

Post-critically finite rational maps

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The eigenvalue of $D_z f$ is called the eigenvalue of f at z and we denote this value by λ_z .

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Critical portrait: $0 \curvearrowright \infty \curvearrowright$

$PC(f) = \{0, \infty\}$. $Fix(f) = \{0, 1, \infty\}$. $\lambda_0 = \lambda_\infty = 0, \lambda_1 = 2$.

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Theorem

Let f be a PCF endomorphism of \mathbb{CP}^1 and let z be a fixed point of f . Then either $\lambda_z = 0$ or $|\lambda_z| > 1$.

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Question

Can we conclude that either $\lambda = 0$ or $|\lambda| > 1$?

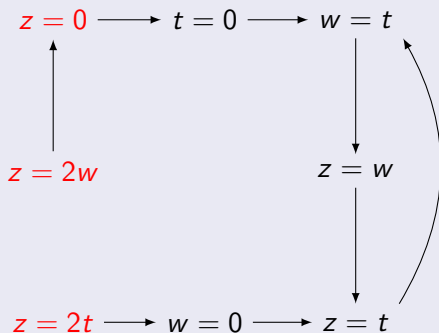
Examples

$$f_1([z_0 : \dots : z_n]) = [z_0^d : \dots : z_n^d], d \geq 2$$

$$PC(f) = \bigcup_{j=1}^n \{[z_0 : \dots : z_n] \mid z_j = 0\}$$

$Fix(f) = \{[\iota_0 : \dots : \iota_n] \mid \iota_j \in \{0, 1\}\}$. The eigenvalues of $D_{z_0} f$ at a fixed point z_0 are 0 and $d \geq 2$.

$$f_2([z : w : t]) = [(z - 2w)^2 : (z - 2t)^2 : z^2]$$



The point $z_0 = [1 : 1 : 1]$ is a fixed point and $D_{z_0} f_2$ has only one eigenvalue -4 of multiplicities 2.

Main results

Theorem (L. ,2019)

Let f be a PCA endomorphism of $\mathbb{C}\mathbb{P}^n$ of degree $d \geq 2$, let z_0 be a fixed point of f and let λ be an eigenvalue of $D_{z_0}f$. If $z_0 \notin PC(f)$ then $|\lambda| > 1$.

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Conjecture is proved in dimension 2!!

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Astorg (2018): The irreducible components of the post-critical set are weakly transverse.

Sketch of proof

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Main cases

- The point z_0 is outside $PC(f)$.
- The point z_0 is inside $PC(f)$.
 - The point z_0 is the regular point of $PC(f)$.
 - The point z_0 is the singular point of $PC(f)$.

The fixed point is outside $PC(f)$

Let f be a PCA endomorphism, z_0 be a fixed point of f and λ be an eigenvalue of f at z_0 . Denote by $X = \mathbb{C}\mathbb{P}^2 \setminus PC(f)$ the complement of $PC(f)$ in $\mathbb{C}\mathbb{P}^2$.

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By study the restriction of a PCA endomorphism on an invariant curve and using the result of PCF endomorphisms, we can prove that either $\lambda = 0$ or $|\lambda| > 1$.

The end!

Thank you for your attention!