Fixed points of post-critically algebraic endomorphisms

Van Tu LE
Institute de Mathématiques de Toulouse

March 25, 2019

Motivation

Post-critically finite rational maps

Let f be an endomorphism of $\mathbb{CP}^1$. The map f is called *post-critically finite* (PCF) if every critical point has finite forward orbit

Motivation

Post-critically finite rational maps

Let f be an endomorphism of $\mathbb{CP}^1$. The map f is called *post-critically finite* (PCF) if every critical point has finite forward orbit

Let C_f be the set of critical points, then f is PCF if the post-critical set $PC(f) = \bigcup_{j \geq 1} f^{\circ j}(C_f)$ is a finite set.

Motivation

Post-critically finite rational maps

Let f be an endomorphism of $\mathbb{CP}^1$. The map f is called *post-critically finite* (PCF) if every critical point has finite forward orbit

Let C_f be the set of critical points, then f is PCF if the post-critical set $PC(f) = \bigcup_{j \geq 1} f^{\circ j}(C_f)$ is a finite set.

Examples

$f(z) = z^2, f(z) = z^2 - 2, f(z) = z^2 + i$

Motivation

Post-critically finite rational maps

Let f be an endomorphism of $\mathbb{CP}^1$. The map f is called *post-critically finite* (PCF) if every critical point has finite forward orbit

Let C_f be the set of critical points, then f is PCF if the post-critical set $PC(f) = \bigcup_{j \geq 1} f^{\circ j}(C_f)$ is a finite set.

Examples

$f(z) = z^2, f(z) = z^2 - 2, f(z) = z^2 + i$

The eigenvalue of $D_z f$ is called the eigenvalue of f at z and we denote this value by λ_z.

$f(z) = z^2$

Critical portrait:
$$\begin{tikzcd} 0 \arrow[loop right] & \infty \arrow[loop right] \end{tikzcd}$$
$PC(f) = \{0, \infty\}$. $Fix(f) = \{0, 1, \infty\}$. $\lambda_0 = \lambda_\infty = 0, \lambda_1 = 2.$

$f(z) = z^2$

Critical portrait: $\begin{tikzcd} 0 \arrow[loop right] & \infty \arrow[loop right] \end{tikzcd}$

$PC(f) = \{0, \infty\}$. $Fix(f) = \{0, 1, \infty\}$. $\lambda_0 = \lambda_\infty = 0, \lambda_1 = 2$.

$f(z) = z^2 + i$

Critical portrait: $\begin{tikzcd} 0 \arrow[r] & i \arrow[r] & -1+i \arrow[l, bend right] & \infty \arrow[loop right] \end{tikzcd}$

$PC(f) = \{i, -1+i, \infty\}$. $Fix(f) = \{\frac{1 \pm \sqrt{1-4i}}{2}, \infty\}$

$\lambda_\infty = 0, \lambda_{\frac{1 \pm \sqrt{1-4i}}{2}} = 1 \pm \sqrt{1-4i}$

$f(z) = z^2$

Critical portrait:
$$\begin{tikzcd} 0 \arrow[loop right] & \infty \arrow[loop right] \end{tikzcd}$$
$PC(f) = \{0, \infty\}$. $Fix(f) = \{0, 1, \infty\}$. $\lambda_0 = \lambda_\infty = 0, \lambda_1 = 2$.

$f(z) = z^2 + i$

Critical portrait:
$$\begin{tikzcd} 0 \arrow[r] & i \arrow[r] & -1+i \arrow[l, bend right] & \infty \arrow[loop right] \end{tikzcd}$$
$PC(f) = \{i, -1+i, \infty\}$. $Fix(f) = \{\frac{1 \pm \sqrt{1-4i}}{2}, \infty\}$
$\lambda_\infty = 0, \lambda_{\frac{1 \pm \sqrt{1-4i}}{2}} = 1 \pm \sqrt{1-4i}$

Theorem

Let f be a PCF endomorphism of $\mathbb{CP}^1$ and let z be a fixed point of f. Then either $\lambda_z = 0$ or $|\lambda_z| > 1$.

Towards higher dimension

Towards higher dimension

Let f be an endomorphism of $\mathbb{CP}^n$. Denote by C_f the set of critical points of f. The post-critical set of f is

$$PC(f) = \bigcup_{j \geq 1} f^{\circ j}(C_f).$$

Towards higher dimension

Let f be an endomorphism of $\mathbb{CP}^n$. Denote by C_f the set of critical points of f. The post-critical set of f is

$$PC(f) = \bigcup_{j \geq 1} f^{\circ j}(C_f).$$

Definition

An endomorphism f of $\mathbb{CP}^n$ is called a *post-critically algebraic* (PCA) if $PC(f)$ is an algebraic set of codim 1 in $\mathbb{CP}^n$.

Towards higher dimension

Let f be an endomorphism of $\mathbb{CP}^n$. Denote by C_f the set of critical points of f. The post-critical set of f is

$$PC(f) = \bigcup_{j \geq 1} f^{\circ j}(C_f).$$

Definition

An endomorphism f of $\mathbb{CP}^n$ is called a *post-critically algebraic* (PCA) if $PC(f)$ is an algebraic set of codim 1 in $\mathbb{CP}^n$.

Let z_0 be a fixed point of f and let λ be an eigenvalue of $D_{z_0} f$.

Towards higher dimension

Let f be an endomorphism of $\mathbb{CP}^n$. Denote by C_f the set of critical points of f. The post-critical set of f is

$$PC(f) = \bigcup_{j \geq 1} f^{\circ j}(C_f).$$

Definition

An endomorphism f of $\mathbb{CP}^n$ is called a *post-critically algebraic* (PCA) if $PC(f)$ is an algebraic set of codim 1 in $\mathbb{CP}^n$.

Let z_0 be a fixed point of f and let λ be an eigenvalue of $D_{z_0} f$.

Question

Can we conclude that either $\lambda = 0$ or $|\lambda| > 1$?

Examples

$f_1([z_0 : \ldots : z_n]) = [z_0^d : \ldots : z_n^d], d \geq 2$

$$PC(f) = \bigcup_{j=1}^{n} \{[z_0 : \ldots : z_n] | z_j = 0\}$$

$Fix(f) = \{[\iota_0 : \ldots : \iota_n] | \iota_j \in \{0, 1\}\}$. The eigenvalues of $D_{z_0} f$ at a fixed point z_0 are 0 and $d \geq 2$.

$f_2([z:w:t]) = [(z-2w)^2 : (z-2t)^2 : z^2]$

$$
\begin{tikzcd}
z=0 \arrow[r] & t=0 \arrow[r] & w=t \arrow[d] \\
z=2w \arrow[u] & & z=w \arrow[d] \\
z=2t \arrow[r] & w=0 \arrow[r] & z=t \arrow[uu, bend right]
\end{tikzcd}
$$

The point $z_0 = [1:1:1]$ is a fixed point and $D_{z_0} f_2$ has only one eigenvalue -4 of multiplicities 2.

Main results

Theorem (L. ,2019)

Let f be a PCA endomorphism of $\mathbb{CP}^n$ of degree $d \geq 2$, let z_0 be a fixed point of f and let λ be an eigenvalue of $D_{z_0}f$. If $z_0 \notin PC(f)$ then $|\lambda| > 1$.

Main results

Theorem (L. ,2019)

Let f be a PCA endomorphism of $\mathbb{CP}^n$ of degree $d \geq 2$, let z_0 be a fixed point of f and let λ be an eigenvalue of $D_{z_0} f$. If $z_0 \notin PC(f)$ then $|\lambda| > 1$.

Theorem (L. ,2019)

Let f be a PCA endomorphism of $\mathbb{CP}^2$ of degree $d \geq 2$ and let z_0 be a fixed point of f. Let λ be an eigenvalue of $D_{z_0} f$. Then either $\lambda = 0$ or $|\lambda| > 1$.

Conjecture

Let f be a PCA endomorphism of $\mathbb{CP}^n$ of degree $d \geq 2$. Let z_0 be a fixed point of f and let λ be an eigenvalue of $D_{z_0} f$. Then either $\lambda = 0$ or $|\lambda| > 1$.

Conjecture

Let f be a PCA endomorphism of $\mathbb{CP}^n$ of degree $d \geq 2$. Let z_0 be a fixed point of f and let λ be an eigenvalue of $D_{z_0} f$. Then either $\lambda = 0$ or $|\lambda| > 1$.

Conjecture is proved in dimension 2!!

Related results

Some geometric or dynamic conditions on the post-critical set and its complement.

Related results

Some geometric or dynamic conditions on the post-critical set and its complement.

Fornæss and Sibony (1994) : The complement of $\overline{PC(f)}$ in $\mathbb{CP}^n$ is Kobayashi hyperbolic and hyperbolically embedded.

Related results

Some geometric or dynamic conditions on the post-critical set and its complement.

Fornæss and Sibony (1994) : The complement of $\overline{PC(f)}$ in $\mathbb{CP}^n$ is Kobayashi hyperbolic and hyperbolically embedded.

Jonsson (1998): The irreducible components of the critical locus are preperiodic.

Related results

Some geometric or dynamic conditions on the post-critical set and its complement.

Fornæss and Sibony (1994) : The complement of $\overline{PC(f)}$ in $\mathbb{CP}^n$ is Kobayashi hyperbolic and hyperbolically embedded.

Jonsson (1998): The irreducible components of the critical locus are preperiodic.

Astorg (2018): The irreducible components of the post-critical set are weakly transverse.

Sketch of proof

Theorem (L. ,2019)

Let f be a PCA endomorphism of $\mathbb{CP}^2$ of degree $d \geq 2$ and let z_0 be a fixed point of f. Let λ be an eigenvalue of $D_{z_0}f$. Then either $\lambda = 0$ or $|\lambda| > 1$.

Sketch of proof

Theorem (L. ,2019)

Let f be a PCA endomorphism of $\mathbb{CP}^2$ of degree $d \geq 2$ and let z_0 be a fixed point of f. Let λ be an eigenvalue of $D_{z_0} f$. Then either $\lambda = 0$ or $|\lambda| > 1$.

Main cases

Sketch of proof

Theorem (L. ,2019)

Let f be a PCA endomorphism of $\mathbb{CP}^2$ of degree $d \geq 2$ and let z_0 be a fixed point of f. Let λ be an eigenvalue of $D_{z_0} f$. Then either $\lambda = 0$ or $|\lambda| > 1$.

Main cases

- The point z_0 is outside $PC(f)$.

Sketch of proof

Theorem (L. ,2019)

Let f be a PCA endomorphism of $\mathbb{CP}^2$ of degree $d \geq 2$ and let z_0 be a fixed point of f. Let λ be an eigenvalue of $D_{z_0}f$. Then either $\lambda = 0$ or $|\lambda| > 1$.

Main cases

- The point z_0 is outside $PC(f)$.
- The point z_0 is inside $PC(f)$.

Sketch of proof

Theorem (L. ,2019)

Let f be a PCA endomorphism of $\mathbb{CP}^2$ of degree $d \geq 2$ and let z_0 be a fixed point of f. Let λ be an eigenvalue of $D_{z_0} f$. Then either $\lambda = 0$ or $|\lambda| > 1$.

Main cases

- The point z_0 is outside $PC(f)$.
- The point z_0 is inside $PC(f)$.
  - The point z_0 is the regular point of $PC(f)$.
  - The point z_0 is the singular point of $PC(f)$.

The fixed point is outside $PC(f)$

Let f be a PCA endomorphism, z_0 be a fixed point of f and λ be an eigenvalue of f at z_0. Denote by $X = \mathbb{CP}^2 \setminus PC(f)$ the complement of $PC(f)$ in $\mathbb{CP}^2$.

The fixed point is outside $PC(f)$

Let f be a PCA endomorphism, z_0 be a fixed point of f and λ be an eigenvalue of f at z_0. Denote by $X = \mathbb{CP}^2 \setminus PC(f)$ the complement of $PC(f)$ in $\mathbb{CP}^2$.

- We consider the universal covering $\pi : \tilde{X} \to X$ of X.

The fixed point is outside $PC(f)$

Let f be a PCA endomorphism, z_0 be a fixed point of f and λ be an eigenvalue of f at z_0. Denote by $X = \mathbb{CP}^2 \setminus PC(f)$ the complement of $PC(f)$ in $\mathbb{CP}^2$.

- We consider the universal covering $\pi : \tilde{X} \to X$ of X.
- We construct a holomorphic map $g : \tilde{X} \to \tilde{X}$ fixing a point w_0 such that

$$\begin{tikzcd}
(\tilde{X}, w_0) \arrow[d, "\pi"'] & (\tilde{X}, w_0) \arrow[l, "g"'] \arrow[d, "\pi"] \\
(X, z_0) \arrow[r, "f"] & (X, z_0)
\end{tikzcd}$$

The fixed point is outside $PC(f)$

Let f be a PCA endomorphism, z_0 be a fixed point of f and λ be an eigenvalue of f at z_0. Denote by $X = \mathbb{CP}^2 \setminus PC(f)$ the complement of $PC(f)$ in $\mathbb{CP}^2$.

- We consider the universal covering $\pi : \tilde{X} \to X$ of X.
- We construct a holomorphic map $g : \tilde{X} \to \tilde{X}$ fixing a point w_0 such that

$$\begin{tikzcd}
(\tilde{X}, w_0) \arrow[d, "\pi"'] & (\tilde{X}, w_0) \arrow[l, "g"'] \arrow[d, "\pi"] \\
(X, z_0) \arrow[r, "f"] & (X, z_0)
\end{tikzcd}$$

- We prove that $\{g^{\circ j}\}_j$ is normal and we use that to construct a center manifold M of g at w_0.

The fixed point is outside $PC(f)$

Let f be a PCA endomorphism, z_0 be a fixed point of f and λ be an eigenvalue of f at z_0. Denote by $X = \mathbb{CP}^2 \setminus PC(f)$ the complement of $PC(f)$ in $\mathbb{CP}^2$.

- We consider the universal covering $\pi : \tilde{X} \to X$ of X.
- We construct a holomorphic map $g : \tilde{X} \to \tilde{X}$ fixing a point w_0 such that

$$\begin{array}{ccc} (\tilde{X}, w_0) & \xleftarrow{g} & (\tilde{X}, w_0) \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ (X, z_0) & \xrightarrow{f} & (X, z_0) \end{array}$$

- We prove that $\{g^{\circ j}\}_j$ is normal and we use that to construct a center manifold M of g at w_0.
- We prove that $g|_M$ is linearizable and use the algebraicity of $PC(f)$ to deduce a contradiction.

The fixed point is outside $PC(f)$

Let f be a PCA endomorphism, z_0 be a fixed point of f and λ be an eigenvalue of f at z_0. Denote by $X = \mathbb{CP}^2 \setminus PC(f)$ the complement of $PC(f)$ in $\mathbb{CP}^2$.

- We consider the universal covering $\pi : \tilde{X} \to X$ of X.
- We construct a holomorphic map $g : \tilde{X} \to \tilde{X}$ fixing a point w_0 such that

$$\begin{tikzcd}
(\tilde{X}, w_0) \arrow[d, "\pi"'] & (\tilde{X}, w_0) \arrow[l, "g"'] \arrow[d, "\pi"] \\
(X, z_0) \arrow[r, "f"] & (X, z_0)
\end{tikzcd}$$

- We prove that $\{g^{\circ j}\}_j$ is normal and we use that to construct a center manifold M of g at w_0.
- We prove that $g|_M$ is linearizable and use the algebraicity of $PC(f)$ to deduce a contradiction.

The fixed point is a singular point

A remarkable case: The fixed point z_0 is a singular point of an invariant irreducible component of $PC(f)$.

The fixed point is a singular point

A remarkable case: The fixed point z_0 is a singular point of an invariant irreducible component of $PC(f)$.

A local situation

Let Γ be a singular germ of of curve of $\mathbb{C}^2$ at $\mathbf{0}$. Denote by m, n the first two Puiseux characteristics of Γ. They are analytic invariants of Γ.

The fixed point is a singular point

A remarkable case: The fixed point z_0 is a singular point of an invariant irreducible component of $PC(f)$.

A local situation

Let Γ be a singular germ of of curve of $\mathbb{C}^2$ at $\mathbf{0}$. Denote by m, n the first two Puiseux characteristics of Γ. They are analytic invariants of Γ.
Let $f : (\mathbb{C}^2, \mathbf{0}) \to (\mathbb{C}^2, \mathbf{0})$ be a finite holomorphic germ fixing Γ. Denote by λ the eigenvalue of a holomorphic germ $\hat{f} : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ which is uniquely determined by f and Γ.

The fixed point is a singular point

A remarkable case: The fixed point z_0 is a singular point of an invariant irreducible component of $PC(f)$.

A local situation

Let Γ be a singular germ of of curve of $\mathbb{C}^2$ at $\mathbf{0}$. Denote by m, n the first two Puiseux characteristics of Γ. They are analytic invariants of Γ.
Let $f : (\mathbb{C}^2, \mathbf{0}) \to (\mathbb{C}^2, \mathbf{0})$ be a finite holomorphic germ fixing Γ. Denote by λ the eigenvalue of a holomorphic germ $\hat{f} : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ which is uniquely determined by f and Γ.
Then λ^m, λ^n are eigenvalues of $D_{\mathbf{0}} f$.

The fixed point is a singular point

A remarkable case: The fixed point z_0 is a singular point of an invariant irreducible component of $PC(f)$.

A local situation

Let Γ be a singular germ of of curve of $\mathbb{C}^2$ at $\mathbf{0}$. Denote by m, n the first two Puiseux characteristics of Γ. They are analytic invariants of Γ.
Let $f : (\mathbb{C}^2, \mathbf{0}) \to (\mathbb{C}^2, \mathbf{0})$ be a finite holomorphic germ fixing Γ. Denote by λ the eigenvalue of a holomorphic germ $\hat{f} : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ which is uniquely determined by f and Γ.
Then λ^m, λ^n are eigenvalues of $D_{\mathbf{0}} f$.

By study the restriction of a PCA endomorphism on an invariant curve and using the result of PCF endomorphisms, we can prove that either $\lambda = 0$ or $|\lambda| > 1$.

The end!

Thank you for your attention!