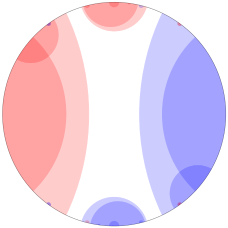
Semigroups of hyperbolic isometries and their parameter spaces



Matthew Jacques

Möbius transformations and hyperbolic geometry

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- When endowed with the supremum metric, the group $\mathcal{M}$ becomes both a complete metric space and a topological group.
- When $\mathbb{D}$ is equipped with the hyperbolic metric ρ, the group $\mathcal{M}$ is exactly the group of orientation-preserving isometries of $(\mathbb{D}, \rho)$, and the unit circle $\mathbb{S}^1$ is its ideal boundary.

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Theorem (Fried, Marotta, Stankewitz, 2012)

If S is not elementary, then $\Lambda^+(S)$ is forward invariant: $f(\Lambda^+(S)) \subseteq \Lambda^+(S)$ for all $f \in S$. Moreover $\Lambda^+(S)$ is the closure of the attracting fixed points of S.

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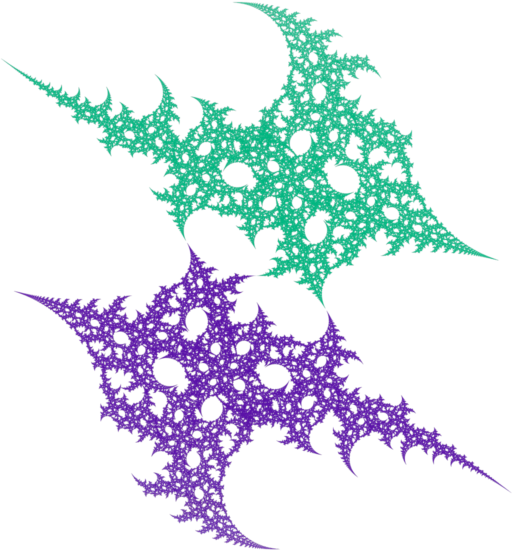
Examples: Fuchsian groups, S conjugate to $\left\langle z \mapsto \frac{1}{3}z,\, z \mapsto \frac{1}{3}z + \frac{2}{3} \right\rangle$.



$\Lambda^+(S)$ and $\Lambda^-(S)$ where $S = \left\langle z \mapsto \frac{a}{1+z},\ z \mapsto \frac{a-1+2ia^{1/2}}{1+z},\ z \mapsto \frac{1}{4(1+z)} \right\rangle$, $a = -0.1 + 0.7i$.

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The limit set $\Lambda(G)$ of a group G of complex Möbius transformations is the subset of $\widehat{\mathbb{C}}$ upon which G is not a normal family.

Parameter spaces of dynamical systems

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The celebrated Mandelbrot set is the collection of those $c \in \mathbb{C}$ such that $J(f_c)$ is connected.

Parameter spaces of semigroups contained in $\mathcal{M}$

Let us take an integer $d > 1$ and endow $\mathcal{M}^d$, the set of d-tuples of elements in $\mathcal{M}$, with the product metric inherited from $\mathcal{M}$.

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Our goal is to classify points in parameter space according to some kind of dynamical behaviour.
To do this we shall use composition sequences and their rates of escape.

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For clarity, let us take $d = 2$ and suppose $\boldsymbol{x} = (g, h)$.

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So by the triangle inequality we have

$$\rho(F_n(0), 0) \leqslant n \max\{\rho(g(0), 0), \rho(h(0), 0)\}.$$

Definition

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Theorem (Yoccoz, 2004)

Both $\mathcal{H}$ and $\mathcal{E}$ are open in $\mathcal{M}^d$. Moreover, $\overline{\mathcal{E}} = \mathcal{M}^d \setminus \mathcal{H}$.

Question 1 (Avila, Bochi, Yoccoz, 2010)

We have $\overline{\mathcal{E}} = \mathcal{M}^d \setminus \mathcal{H}$, is it true that $\overline{\mathcal{H}} = \mathcal{M}^d \setminus \mathcal{E}$?

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For each $d > 2$ the answer to Question 1 is no[1], we can find points that lie in $\mathcal{M}^d \setminus \mathcal{E}$ but not $\overline{\mathcal{H}}$.

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All our counter examples to Question 1 correspond to discrete groups, and so contain the identity.

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- $\Lambda^+ : \mathcal{M}^d \to \mathcal{C}$ as the function given by $\boldsymbol{x} \mapsto \Lambda^+(S(\boldsymbol{x}))$; and similarly
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The hyperbolic locus is exactly the set of those $\boldsymbol{x} \in \mathcal{M}^d$ for which

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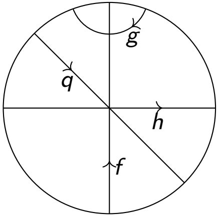
Let $\boldsymbol{y} = (f, g, h, q) \in \mathcal{M}^4$, where f, h, q generate (as a group) a Schottky group, and $g = fh^{-1}f^{-1}$.



For each $x \in \mathcal{M}^d$ we define

$$\chi(\boldsymbol{x}) = \text{ no. of complementary components of } \Lambda^-(\boldsymbol{x}) \text{ that meet } \Lambda^+(\boldsymbol{x}).$$

Then $\chi(\boldsymbol{y}) = +\infty$.

Thank you for your attention!