Local Random Dynamics

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March 25, 2019

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We want to classify stability in term of the κ_i.

Classification of stability

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Neutral germ $\kappa_1 = 0$ and $s = 1$: then f is stable iff is conjugate to the linear map Uz, with U unitary matrix;

Semi-attracting germ $\kappa_1 = 0$ and $\kappa_2 < 0$: when $m = 2$ then f is stable iff is conjugate to

$$\widetilde{f}(x, y) = (\lambda x, \mu y + O(x)y),$$

where $|\lambda| = 1$ and $|\mu| < 1$ are the two eigenvalues of $df(0)$.

Random dynamics

Let ν be a probability measure on $\mathcal{O}(\mathbb{C}^m, 0)$ with compact support. Write Ω for the space of all sequences in *supp*(ν) and $\mu = \nu^\infty$. Let $\omega = (f_{\omega_1}, f_{\omega_2}, \dots) \in \Omega$ and write

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By ergodicity of the shift map T, the family $\{f_\omega^n\}_{n\in\mathbb{N}}$ is normal with probability 0 or 1.

Definition

A probability measure ν is *a.s. stable* if $\{f_\omega^n\}_{n\in\mathbb{N}}$ is normal with probability 1, and *a.s. unstable* if is normal with probability 0.

Lyapunov exponent

Theorem (Multiplicative ergodic theorem)

There exists numbers $+\infty > \kappa_1 > \cdots > \kappa_s$ called Lyapunov exponents so that for almost every $\omega \in \Omega$ we can find a collection of subspaces $V_1 \supset \ldots V_s \supset V_{s+1}$, with $V_1 = \mathbb{C}^m$ and $V_{s+1} = \{0\}$, and so that for every $v \in V_i \setminus V_{i+1}$ we have

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$$\lim_{n\to\infty} n^{-1} \log \|df^n_\omega(0)v\| = \kappa_i.$$

Lyapunov exponents can be use to classify the a.s. stability of the measure ν.

Classification of stability (Easy cases)

Suppose that ν is an:

Attracting measure $\kappa_1 < 0$: then ν is a.s stable;

Repelling measure $\kappa_1 > 0$: then ν is a.s unstable.

If *supp*(ν) is not compact then this is not true in general (even the Multiplicative Ergodic Theorem requires some additional assumption).

We will now analyze a.s. stability for **Neutral measures** ($\kappa_1 = 0$ and $s = 1$) and **Semi-attracting measures** ($\kappa_1 = 0$ and $\kappa_2 < 0$).

Neutral measures

Write $S_\nu = \{f_\omega^n : \omega \in \Omega, n \in \mathbb{N}\}$ for the semigroup generated by $supp(\nu)$.

Theorem

A neutral measure is a.s. stable iff all the germs in $supp(\nu)$ are simultaneously linearizable, and the semigroup of differentials

$$dS_\nu = \{df_\omega^n(0) : \omega \in \Omega,\ n \in \mathbb{N}\},$$

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Corollary

A neutral measure is a.s. stable if and only if the origin lies in the Fatou set of the semigroup S_ν.

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Step 2: Given $\omega \in \Omega$ for which $\{f_\omega^n\}_{n\in\mathbb{N}}$ is normal, Using Hurwitz theorem we find $\varepsilon, \delta > 0$ and $\rho > 0$ so that

$$B(0,\rho) \subset f_\omega^n(B(0,\delta)) \subset B(0,\varepsilon), \qquad \forall n \geq 0.$$

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Step 3: For a.e ω it holds that for every $n \in \mathbb{N}$ and $\sigma \in \Omega$ the iterate f_σ^n belongs to the closure of $\{f^n_{T^N\omega}\}_{N\in\mathbb{N}}$.

Step 4: Choose ω as in Step 3 and so that $\{f_\omega^n\}_{n\in\mathbb{N}}$ is a normal family. Choose ε, δ and ρ as in Step 2. Then for every $\sigma \in \Omega$ and $n \in \mathbb{N}$ we have

$$f_\sigma^n(B(0,\rho)) \subset f_{T^N\omega}^n(B(0,\rho)) \subset f_\omega^n(B(0,\delta)) \subset B(0,\varepsilon).$$

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Step 5: Conclude by Bochner's linearization Theorem.

Semi-attracting measures

We consider only the case $m = 2$. It is not possible to classify a.s. stability only in term of linearizability of the elements in $supp(\nu)$.

Example

Consider the semi-attracting measure $\nu = \delta_f/2 + \delta_g/2$, where

$$f(z,w) = (z, w/2),$$
$$g(z,w) = (z + zw, w).$$

This measure is a.s. stable but the germ g is parabolic and therefore not stable.

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Proposition

For every $\omega \in \Omega$ and for every $n \in \mathbb{N}$ we have

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Theorem (Stable manifold Theorem)

Let $\omega \in \Omega$ so that $\{f_\omega^n\}_{n\in\mathbb{N}}$ is a normal family. Then the image of every limit map g of a subsequence $f_\omega^{n_k}$ is locally a one dimensional manifold Σ_g. Furthermore given z sufficiently close to 0 its stable set

$$W_\omega^s(z) := \{w \ : \ |f_\omega^n(w) - f_\omega^n(z)| \to 0\},$$

is locally a one dimensional manifold.

What next?

We say that the measure ν is strongly irreducible if there are no proper subspaces which are invariant for $df(0)$ for every germ $f \in supp(\nu)$.

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Is there a semi-attracting measure ν which is a.s. stable and strongly irreducible?

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Question 2

Do Σ_g and $W^s_\omega(z)$ always depend on the limit map g and ω?