A new family of functions without wandering domains

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Basic definitions

Let $f : \mathbb{C} \to \mathbb{C}$ be meromorphic.

The **Fatou set**, $F(f)$, is the set of points for which there is a neighbourhood where the family of iterates is equicontinuous.

The **Julia set**, $J(f)$, is the complement of the Fatou set.

The **escaping set**, $I(f)$, is the set of points that tend to infinity under iteration.

Let U be a Fatou component of f such that, for any $n, m \in \mathbb{N}$ with $n \neq m$, $f^n(U) \cap f^m(U) = \varnothing$. Then U is called a **wandering domain** for f. In principle, U can be either *escaping*, *oscillating*, or *of bounded orbit.*

Wandering domains

Sullivan (1985): No wandering domains for rational functions.

Examples for wandering domains of transcendental entire functions include **escaping** ones by Baker (1976), Herman (1982), Bergweiler (2007) among others; and **oscillating** ones by Eremenko and Lyubich (1987), Bishop (2015), Martí–Pete and Shishikura (2018) among others. Note that the existence of wandering domains **of bounded orbit** is a major open question.

Classes of transcendental entire functions with **no wandering domains** have been found by Eremenko and Lyubich (1984), Goldberg and Keen (1986), Stallard (1991), Bergweiler (1993), Mihaljević–Brandt and Rempe–Gillen (2013) among others.

Our result

We define the family of transcendental entire functions

$$\mathcal{F} = \left\{ f : f(z) = \sum_{k=0}^{p-1} \exp(\omega_p^k z) \text{ for any } p \geq 3 \right\},$$

where $\omega_p = \exp(2\pi i/p)$ is a pth root of unity.

We prove the following:

Theorem

Let $f \in \mathcal{F}$ with p even. Then f has no wandering domains.

Main tool

Theorem (Barański, Fagella, Jarque, Karpińska; 2017)

Let f be a transcendental meromorphic map and U be a Fatou component of f. Denote by U_n the Fatou component such that $f^n(U) \subset U_n$. Then for every $z \in U$ there exists a sequence $p_n \in P(f)$ such that

$$\frac{\operatorname{dist}(p_n, U_n)}{\operatorname{dist}(f^n(z), \partial U_n)} \to 0, \text{ as } n \to \infty.$$

In particular, if for some $d > 0$ we have $\operatorname{dist}(f^n(z), \partial U_n) < d$ for all n, then $\operatorname{dist}(p_n, U_n) \to 0$ as n tends to ∞.

Lemma

Let $f \in \mathcal{F}$. Then there exists $M > 0$ such that $\operatorname{dist}(z, J(f))$ is bounded for all $z \in F(f)$.

The postsingular set

The point z is a *critical point* of f if $f'(z) = 0$. If z is a critical point, then $f(z)$ is called a *critical value* of f.

A (finite) *asymptotic value* of f, is a value $w \in \mathbb{C}$ for which there exists a curve $\gamma : (0, \infty) \to \mathbb{C}$ with $\gamma(t) \to \infty$ as $t \to \infty$, such that $f(\gamma(t)) \to w$ as $t \to \infty$.

We denote the critical values and finite asymptotic values of f as $CV(f)$ and $AV(f)$ respectively. We define the set of *singular values* of f as

$$S(f) := \overline{CV(f) \cup AV(f)}.$$

We further define the *postsingular set* as

$$P(f) := \overline{\cup_{n \geq 0} f^n(S(f))}.$$

Lemma

Let $f \in \mathcal{F}$. Then $P(f) \subset \mathbb{R}$.

Spiders' webs and Cantor bouquets

Definition

A set $E \subset \mathbb{C}$ is called a *spider's web* if it is connected and there exists a sequence of bounded simply connected domains G_n with $G_n \subset G_{n+1}$ for $n \in \mathbb{N}$, $\partial G_n \subset E$ for $n \in \mathbb{N}$, and $\cup_{n \in \mathbb{N}} G_n = \mathbb{C}$.

Definition (informal)

A *Cantor bouquet* is an uncountable collection of pairwise disjoint curves to infinity; roughly speaking $C \times [0, \infty)$, where C is a Cantor set.

Theorem

Let $f \in \mathcal{F}$. Then $J(f)$ is a spider's web (Sixsmith; 2015) that contains a Cantor bouquet.

Extra tools

Theorem (Osborne; 2012)

Let f be a transcendental entire function and let $A_R(f)$ be a spider's web. Then f has no wandering domains of bounded orbit.

Lemma (Sixsmith; 2015)

Suppose that f is a transcendental entire function and that $z_0 \in I(f)$. Set $z_n = f^n(z_0)$, for $n \in \mathbb{N}$. Suppose that there exist $\lambda > 1$ and $N \geq 0$ such that

$$f(z_n) \neq 0 \quad \text{and} \quad \left| z_n \frac{f'(z_n)}{f(z_n)} \right| \geq \lambda, \quad \text{for } n \geq N.$$

Then either z_0 is in a multiply connected Fatou component of f, or $z_0 \in J(f)$.