Transcendental Julia Sets with Fractional Packing Dimension

Jack Burkart, Stony Brook

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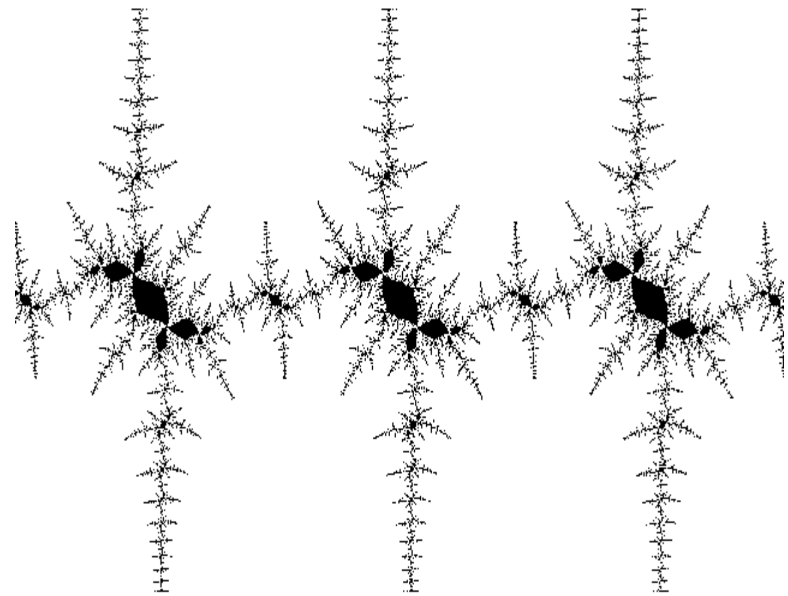
PART I: HISTORY AND DEFINITIONS

Theorem (Baker): Julia sets of transcendental entire functions contain non-degenerate continua. Hausdorff dimension is lower bounded by 1.

Theorem (Misiurewicz): Julia set of $\exp(z) = \mathbb{C}$.

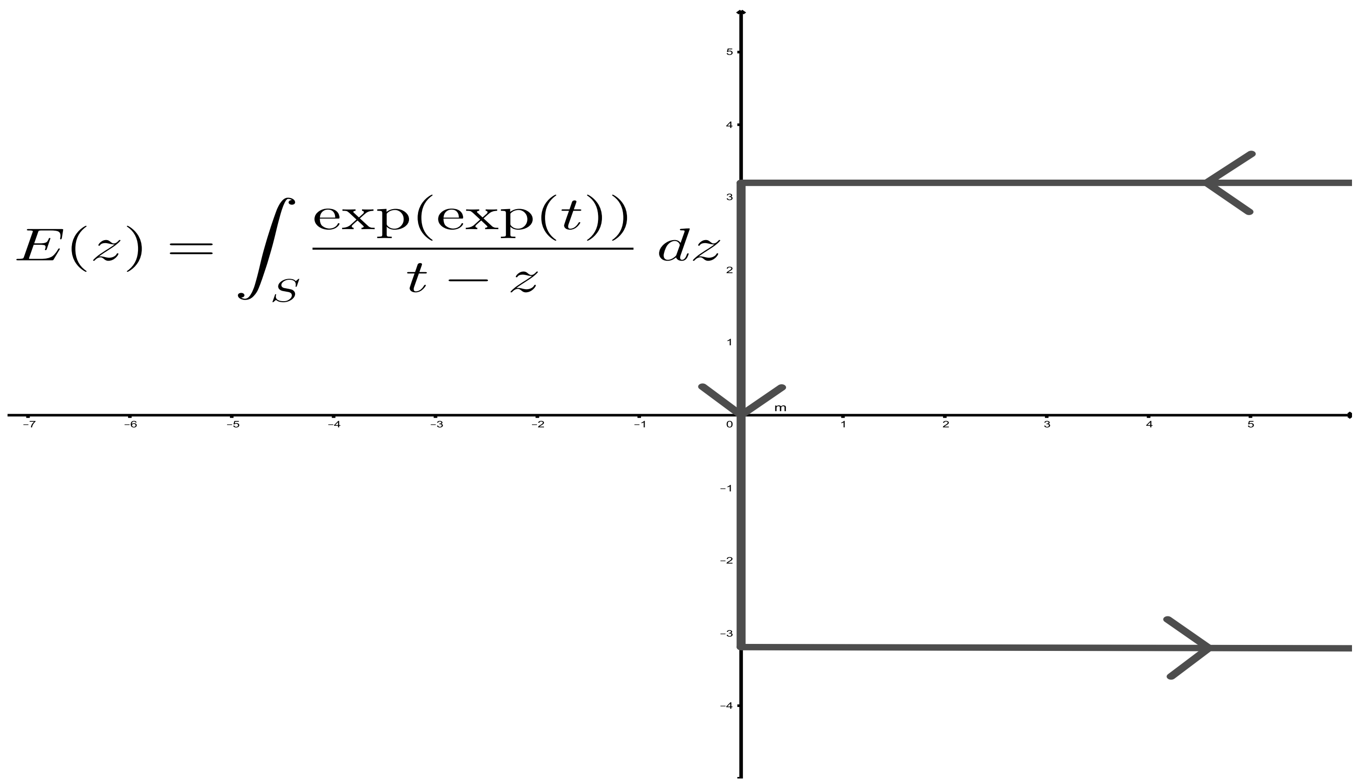
Theorem (McMullen): sine family always has positive area. exp family always has dimension 2. Zero area if there is an attracting cycle.



Julia set in the cosine family.

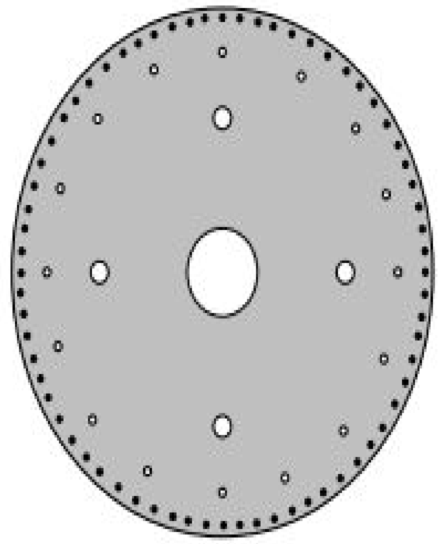
Theorem (Stallard): There exist functions in $\mathcal{B}$ with Julia set with dimension arbitrarily close to 1; dimension 1 does not occur in $\mathcal{B}$. All dimensions in $(1, 2]$ occur in $\mathcal{B}$.

$$E(z) = \int_S \frac{\exp(\exp(t))}{t - z} \, dz$$



$E_K(z) = E(z) - K$. Dimension tends to 1 as K increases

Theorem (Bishop): There exists a transcendental entire function whose Julia set has Hausdorff dimension packing dimension equal to 1.



The functions are of the form

$$f_{\lambda,R,N}(z) = [\lambda(2z^2 - 1)]^{\circ N} \cdot \prod_{k=1}^{\infty} \left(1 - \frac{1}{2}\left(\frac{z}{R_k}\right)^{n_k}\right).$$

There are two other useful notions of dimension for t.e.f's.

The upper Minkowski dimension (we require K compact)

$$\dim_M(K) = \limsup_{\epsilon \to 0} \frac{\log(N(K,\epsilon))}{-\log(\epsilon)}.$$

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Lemma: Let K be a compact set. Then

$$\dim_H(K) \leq \dim_P(K) \leq \dim_M(K).$$

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For our application we have fall all bounded open sets U which intersect the Julia set

$$\dim_H(J(f)) \leq \dim_P(J(f)) \leq \dim_{U,M}(J(f)).$$

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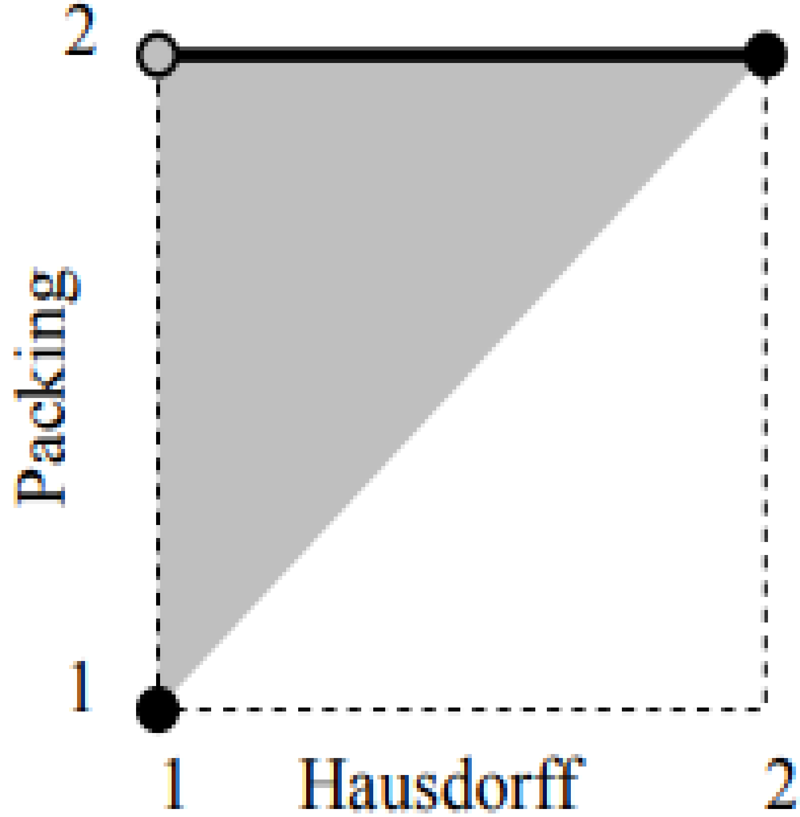

Previous chart of attained dimensions.

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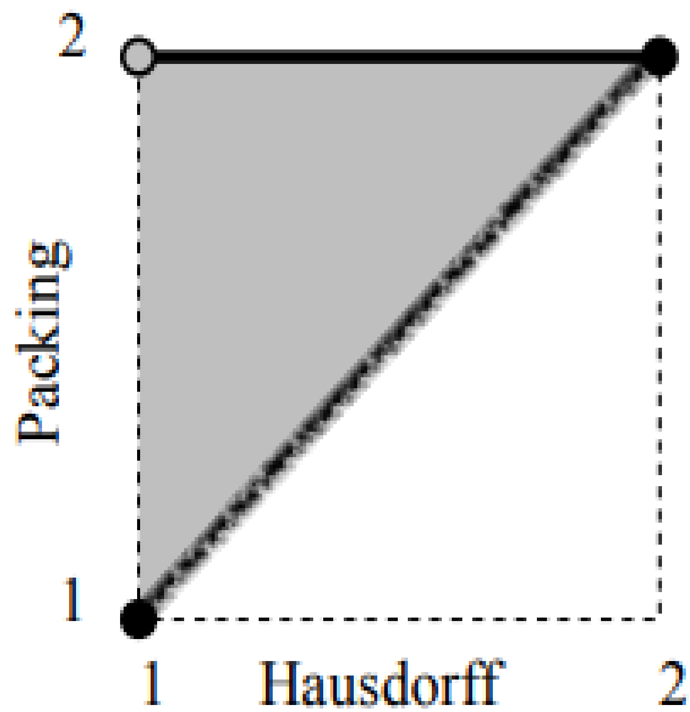

Updated possible dimensions chart.

PART II: Properties of the Function f.

Lemma: The following defines a transcendental entire function:

$$f(z) := (z^2 + c)^{\circ N} \prod_{k=1}^{\infty} \left(1 - \frac{1}{2}\left(\frac{z}{R_k}\right)^{n_k}\right) := f_c^N(z) \prod_{k=0}^{\infty} F_k(z)$$

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1. N is a large integer; $n_k = 2^{N+k-1}$.

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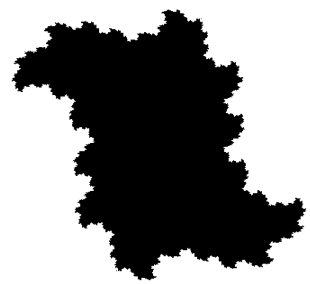
$$f(z) = z^{1024}\left(1 - \frac{1}{2}\left(\frac{z}{20^{1024}}\right)^{1024}\right)\left(1 - \frac{1}{2}\left(\frac{z}{20^{1024} \cdot 2^{2048}}\right)^{2048}\right)\cdots$$

Behavior of f near the origin.

$$f(z) = (z^2 + c)^{\circ N}(1 + \epsilon(z))$$



f is a degree 2^N polynomial-like mapping. Can get a *lower* bound on the Hausdorff dimension of the Julia set of the entire function f by estimating the dimension of the Julia set $\partial K(f)$ of the *polynomial-like* map f.

Theorem: Let $\delta > 0$ be given. Then f may be defined so that

$$|\dim_H(J(f_c)) - \dim_H(\partial K(f))| < \delta.$$

It follows that $\dim_H(J(f)) \geq s - \delta$; the dimension at worst shrinks by a small amount.

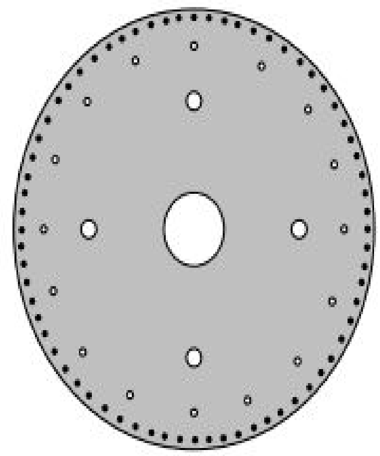
Two proof strategies:

1. Construct a quasiconformal mapping of a neighborhood of the Julia set directly.

2. Introduce a new parameter λ into $\epsilon(z)$. The Julia set moves holomorphically in this case.

Partitioning the Fatou and Julia Set

Schematic of the “Round” Fatou component Ω_k containing $|z| = R_k$, $k \geq 1$.



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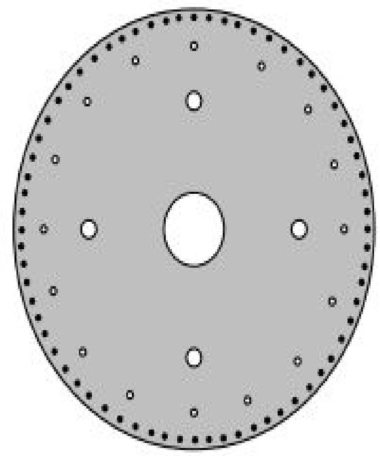

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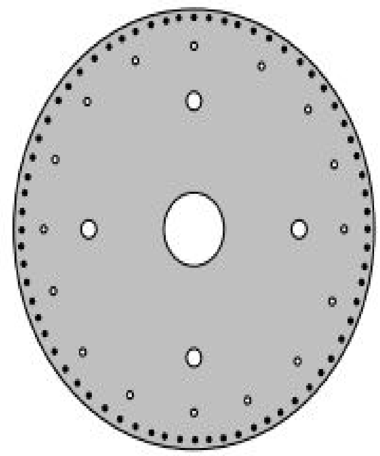

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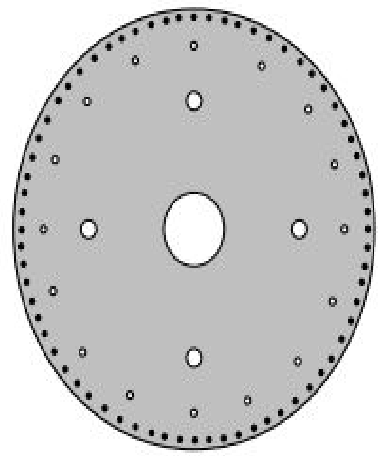

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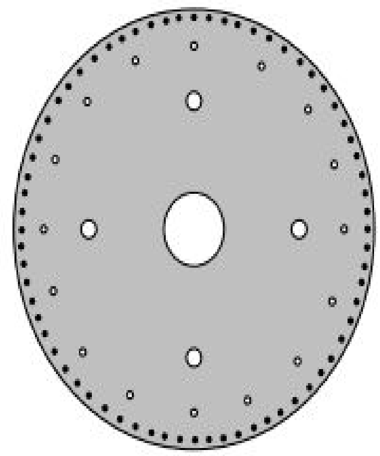
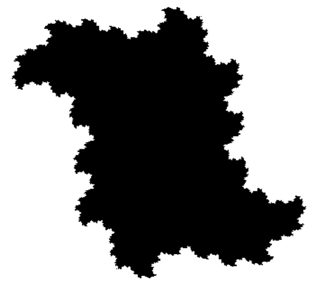


1. The inner and outer boundary curves are C^1 and close to circles.
2. The smaller boundary components are close to circles and arranged in circular layers.
3. All interior and boundary points iterate to Ω_{k+1}.
4. All points in holes iterate "backwards."

Partitioning the Fatou and Julia Set

The basin of attraction B_f of the polynomial-like f



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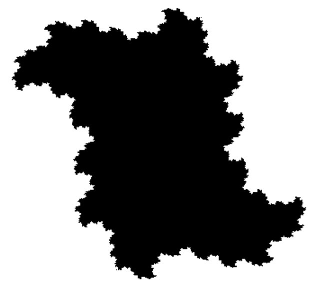

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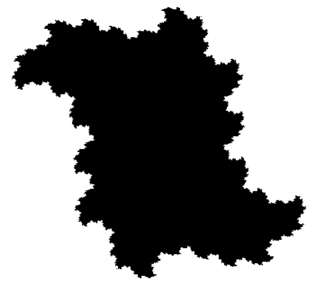

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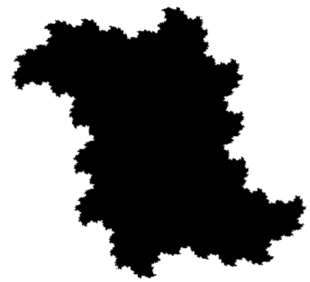
The basin of attraction B_f of the polynomial-like f



1. f is a hyperbolic polynomial-like mapping, so the packing and Hausdorff dimensions coincide
2. Contains the origin; hence all the zeros of f land inside this basin.
3. f behaves like z^{2^N} outside B_f.

Partitioning the Fatou and Julia Set

"Windy" Fatou components $\Omega_{-k+1} = f^{-k}(\Omega_1)$, $k \geq 1$.

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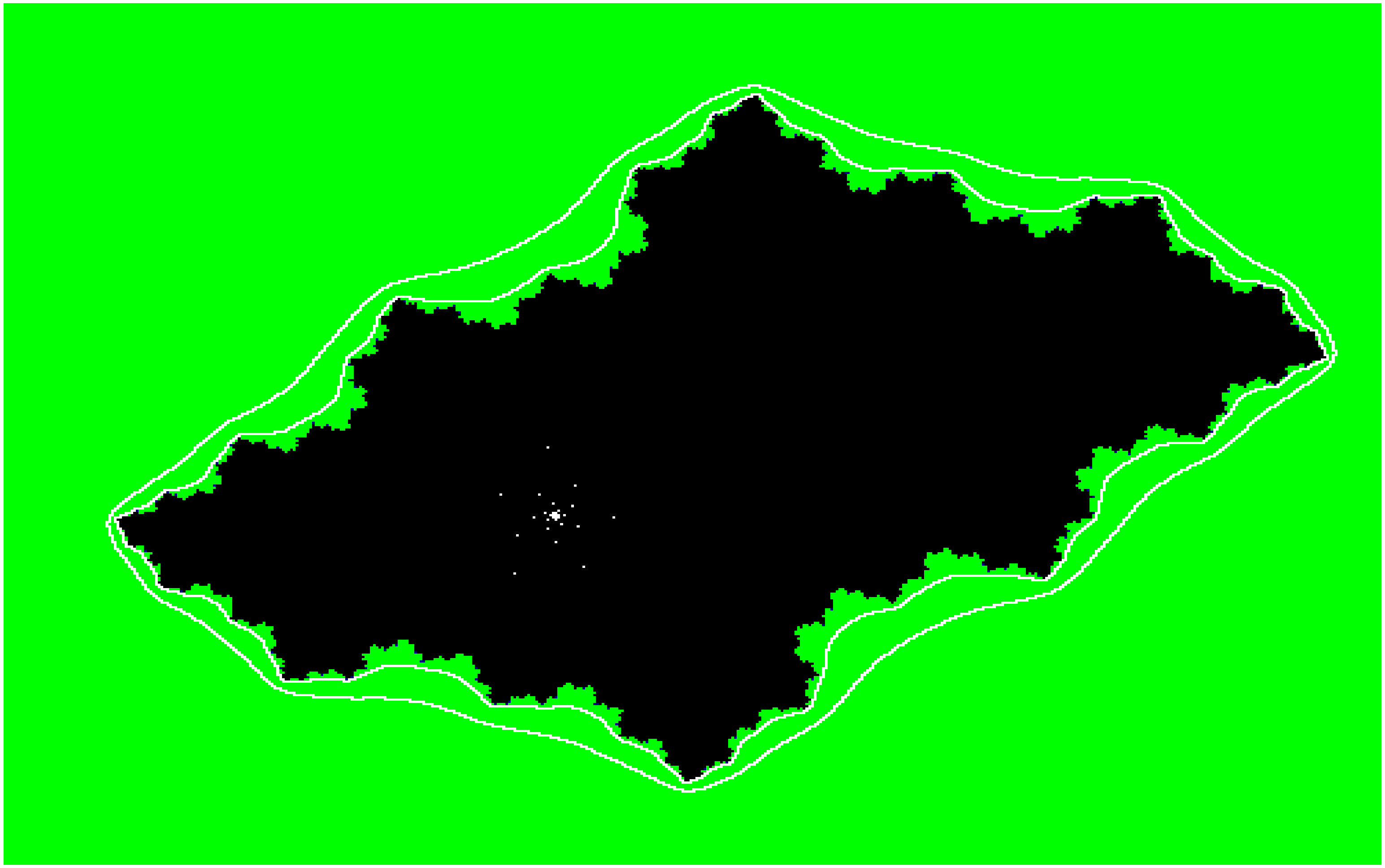

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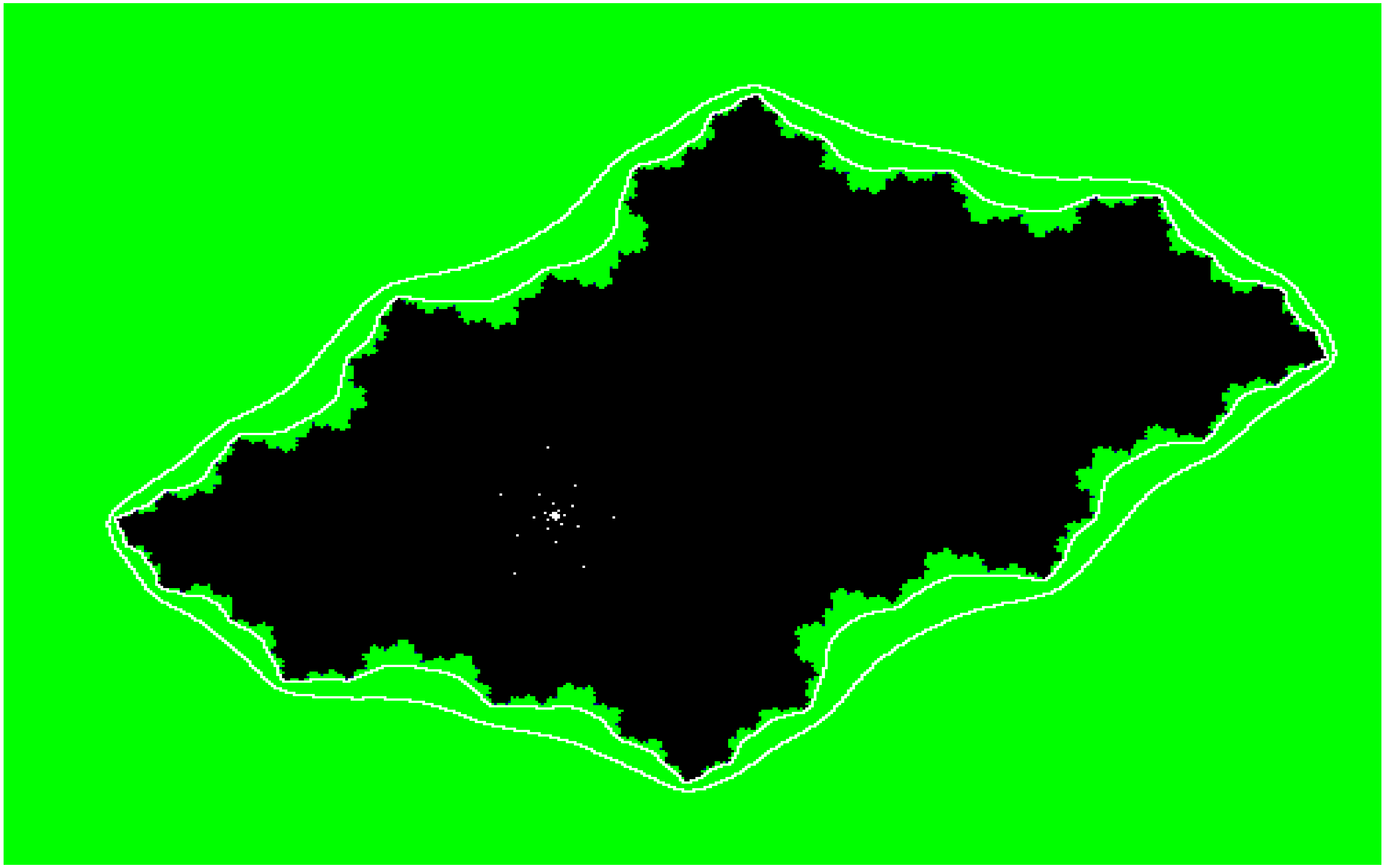
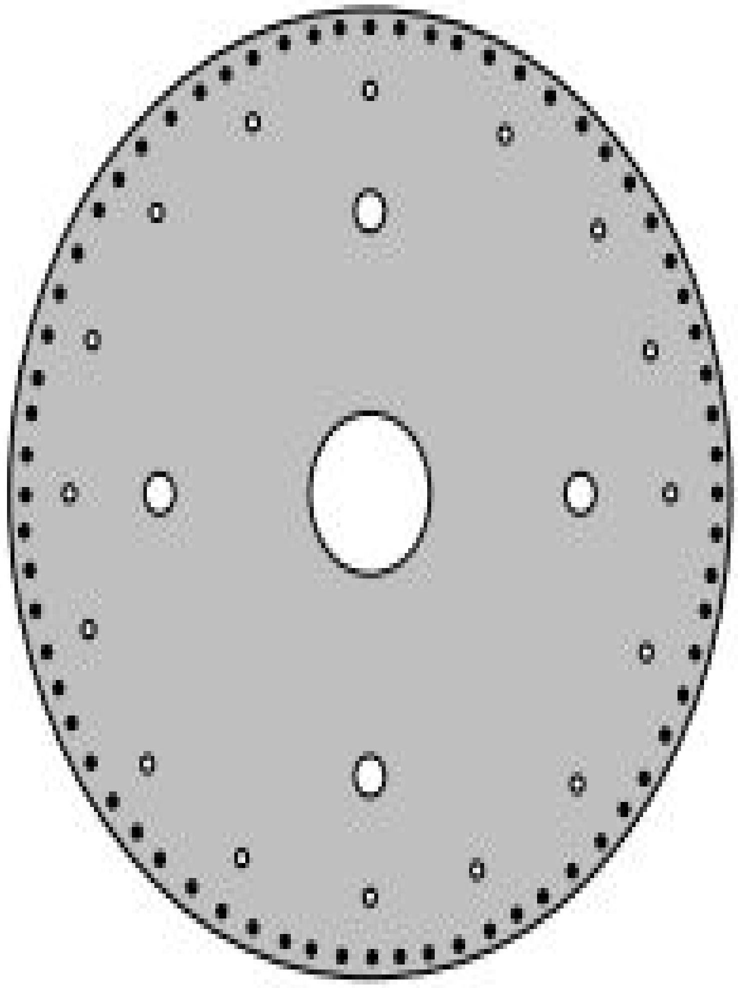


Same topology as round components; new geometry introduced by B_f.

What happens when we zoom into one of the holes?



Its the same picture!

Theorem: Let ω be a Fatou component for f. Then there exists a unique m so that $f^m(\omega)$ is

1. $f^m(\Omega) = \Omega_k$, $k \geq 1$. "A component of Ω_k type." ω is a round component.

2. $f^m(\omega) = \Omega_k$ for $k \leq 0$. ω is a windy component.

3. $f^m(\omega) = B_f$. ω is a copy of the basin of attraction.

Moreover, each component ω is iterated conformally to its category above with bounded conformal distortion.

What about the points in the holes infinitely often?

Theorem: The set Y of points contained in infinitely many holes is the set of buried points in the Julia set. In particular

1. Y is dense in the Julia set.

2. The dimension of Y is at least that of the basin B_f,
$$\dim_H(B_f) \le \dim_H(Y) \le \dim_H(B_f) + \epsilon.$$

3. Y contains the slow escaping set, bounded orbit set, and the bungee set.

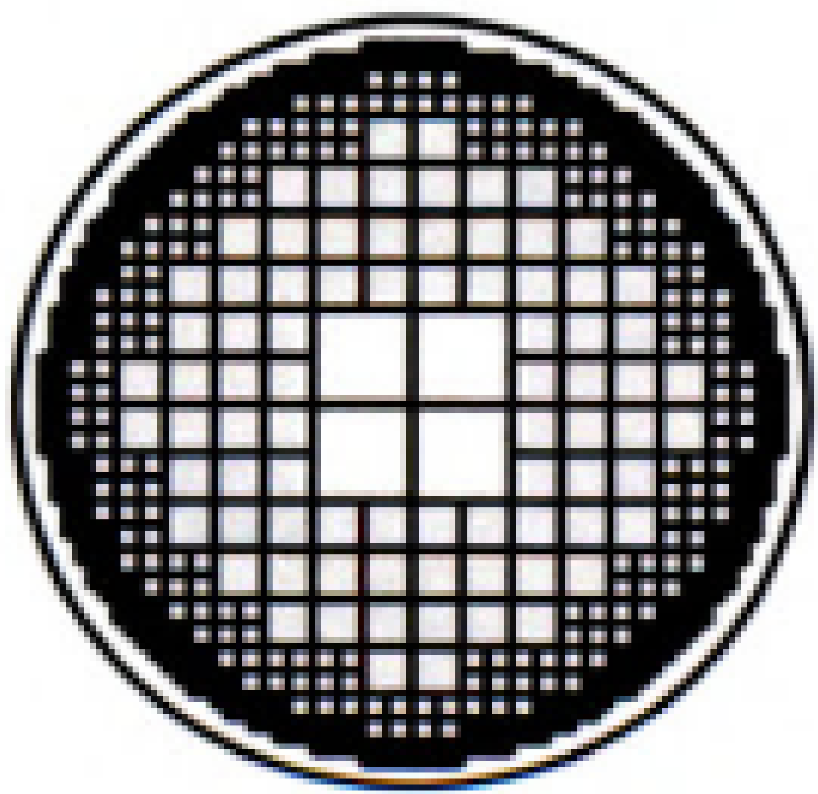
PART III: Controlling the Packing Dimension

Whitney decompositions.

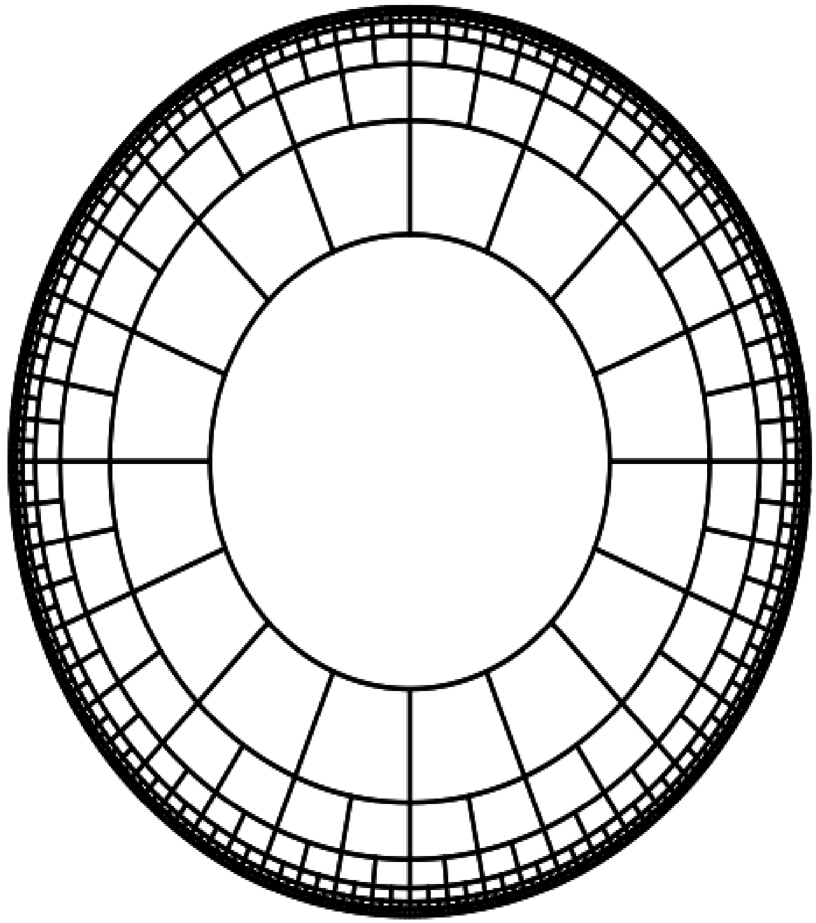
Let Ω be a bounded open set. A Whitney decomposition of Ω into cubes is a collection of open cubes $\{Q_j\}$ satisfying:

1. The cubes have pairwise disjoint interior.
2. $\Omega = \cup \overline{Q_j}$.
3. There exists a constant C so that
$$\frac{1}{C}\mathrm{dist}(Q_j, \partial\Omega) \leq \mathrm{diam}(Q_j) \leq C\mathrm{dist}(Q_j, \partial\Omega)$$

The collection $\{Q_j\}$ need not be literal cubes, so long as the boundaries of the Q_j have zero measure.



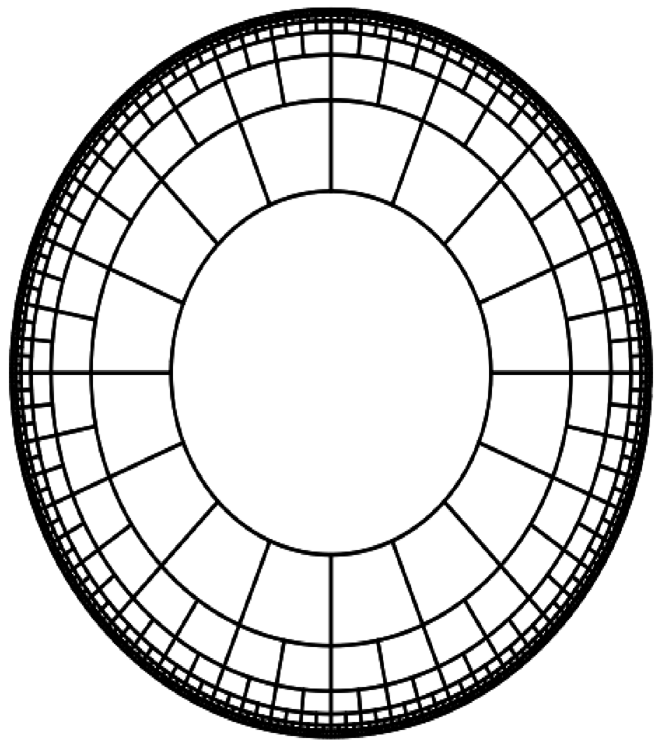
Whitney decomposition of $\mathbb{D}$ with dyadic squares.



Whitney decomposition of $\mathbb{D}$ with hyperbolic squares.

We may define the **critical exponent** of a Whitney decomposition:

$$\alpha(K) = \inf\{\alpha : \sum |Q|^\alpha < \infty\}$$



Example: $\sum |Q|^t \asymp \frac{1}{t-1}\mathrm{diam}(\mathbb{D})^t$

The key idea is that we may connect the upper Minkowski dimension to the critical exponent of Whitney decompositions.

Theorem: Let K be a compact set with zero Lebesgue measure. Then

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Recall that by the results of Rippon and Stallard, to compute the packing dimension, it suffices to compute $\dim_{M,B}(J(f))$, where B is a ball containing Ω_1. We will do this by the lemma above.

Theorem: Let $\epsilon > 0$ be given. Then f may be defined so that the critical exponent satisfies

$$|\alpha(J(f) \cap B) - s| < \epsilon.$$

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Proof: We have

$$s - \delta \leq \dim_H(J(f)) \leq \dim_P(J(f)) \leq s + \epsilon.$$

δ and ϵ can both be arranged to be arbitrarily small.

Proof of Theorem:

Start with a Whitney decomposition W of the complement of $J(f)$ inside of the ball B. Let $t > s + \epsilon$. Then

$$\sum_{Q \in W} |Q|^t = \sum_{Q \in W(\text{Basins})} |Q|^t + \sum_{Q \in W(\text{Round})} |Q|^t + \sum_{Q \in W(\text{Windy})} |Q|^t$$

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In Bishop's dimension 1 paper, his estimates work for the cubes in $W(Round)$.

The idea for the other two sums is to transfer the calculation to a canonical region and estimate the errors using conformal mapping estimates.

For example, we may sum over each inverse image of B_f:

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By the Kobe distortion theorem

$$\sum_{Q \in W(B_i)} |Q|^t \le C \cdot \text{diam}(\omega) \cdot \sum_{Q \in W(B_i)} |f^m(Q)|^t.$$

Lemma: There exists a constant C independent of the conformal mapping f^m so that

$$\sum_{Q \in W(B_i)} |f^m(Q)|^t \leq C \sum_{Q \in W(B_f)} |Q|^t.$$

Lemma: The components ω have summable diameter:

$$\sum_{\omega} \mathrm{diam}(\omega)^{s+\epsilon} < \infty.$$

It follows that

$$\sum_{Q \in W(\text{Basins})} \leq C \cdot \sum \mathrm{diam}(\omega)^t \cdot \sum_{Q \in W(B_f)} \mathrm{diam}(Q)^t.$$

Dimension of $\partial B_f < t$, so the sum converges. We use a similar but more involved approach for the windy components.

Thanks for listening! Any questions?

Questions I have

1. Are all packing dimensions in $(1, 2)$ attainable?

2. Can we arrange for $\dim_P(J(f)) = \dim_H(J(f))$? Or is the inequality somehow strict?

3. Is it a lost cause to generate computer images of multiply connected Fatou components?

4. Can we calculate the dimension of $BU(f)$ and $BO(f)$ in these examples? Are they the same as the dimension of $J(f)$?

5. Is it interesting that the Julia set in this examples has C^1 and fractal components?