Puzzle for rational maps

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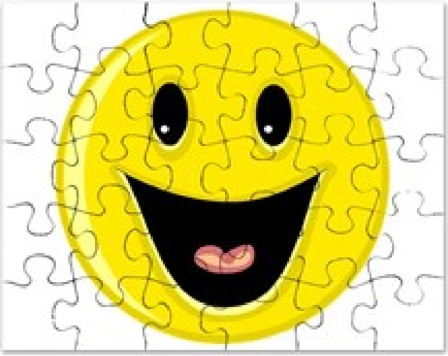

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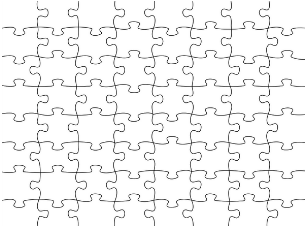
- $$P \in \mathcal{P}_n \implies \exists Q \in \mathcal{P}_{n-1}, \quad P \subset Q$$

  Q is unique. The idea is to get a more complicate jigsaw puzzle at each next level (with more puzzle pieces)



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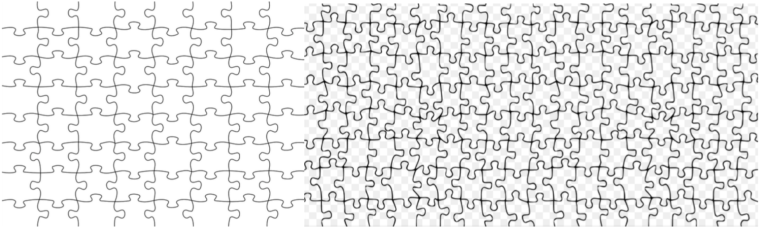

- the map f acts on the puzzle :

$$P \in \mathcal{P}_n \implies f(P) \in \mathcal{P}_{n-1}$$

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On some subset of X, one can define $P_n(x)$ as the unique puzzle piece of level containing x

$$P_0(x) \supset P_1(x) \supset ... \supset P_n(x) \supset P_{n+1}(x) \supset .... \ni x$$

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$$P_0(f(x)) = f(P_1(x)) \supset P_1(f(x)) = f(P_2(x)) \supset ... \supset P_{n-1}(f(x)) = f(P_n(x))$$

is the sequence of decreasing puzzle pieces containing $f(x)$

If the diameter of the sequence of puzzle pieces shrinks to 0

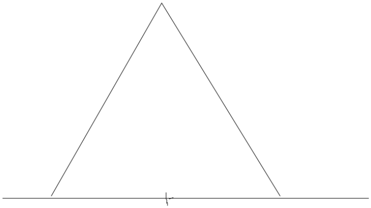
$$P_0(x) \supset P_1(x) \supset ... \supset P_n(x) \supset P_{n+1}(x) \supset .... \ni x$$

then it gives a precise location of the point x.

Example

The tent map is defined on $[0,1]$ by

$$T(x) = \begin{cases} 2x & \text{if} \quad 0 \leq x \leq 1/2 \\ 2(1-x) & \text{if} \quad 1/2 \leq x \leq 1 \end{cases}$$



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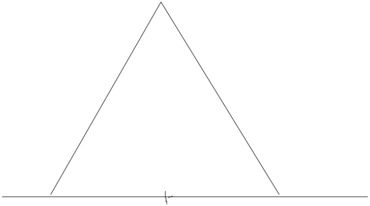

$$\mathcal{P}_0 = \{]0,1[\}, \mathcal{P}_1 = \{]0,1/2[,]1/2,1[\},$$
$$\mathcal{P}_2 = \{]0,1/4[,]1/4,1/2[,]1/2,3/4[,]3/4,1[\}\ldots$$

A basic method of studying its dynamics is to find a symbolic representation: an encoding of the points by sequences of symbols such that the map T becomes the shift map.

Puzzles are the analogous of a Markov partition for hyperbolic systems.

Sinaï and Bowen used Markov partition to describe uniformly hyperbolic systems.

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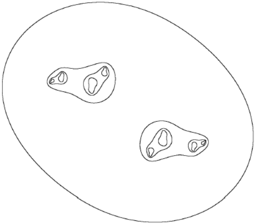


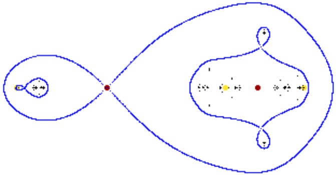
Puzzle pieces are obtained by pull back by the dynamics of an starting partition

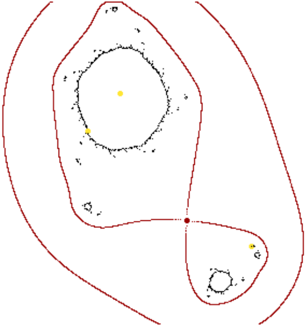
This way of understanding the dynamics using coding was already used in the study of hyperbolic systems.

The presence of dilatation, contraction but also bending does not allow to build a general theory.

It will be more a collection of examples.

Branner-Hubbard puzzle, cubic polynomials





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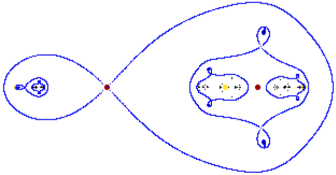

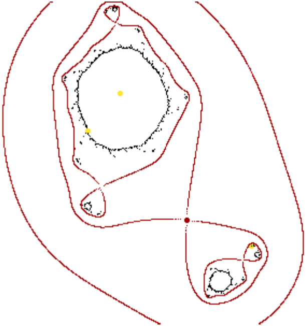

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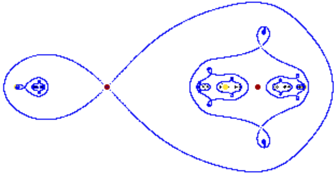

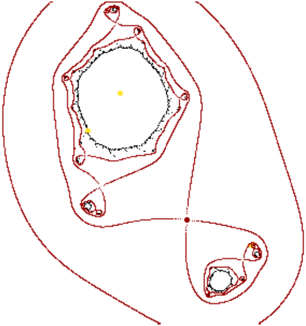

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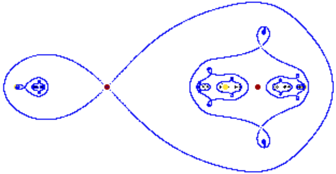

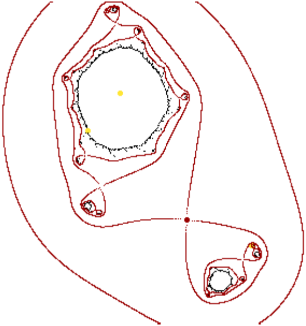

Theorem (B-H)

For a cubic polynomial f with one critical point in $K(f)$, the Julia set $K(f)$ is a Cantor set if and only if the critical components of $K(f)$ are aperiodic.

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- $f(P_{n+1}(x)) = P_n(f(x))$
- $f : P_{n+1}(x) \to P_n(f(x))$ is a covering of degree at most 2

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Every point $x \in K(f)$ defines a "nest" of puzzle pieces

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Remark

- $K(f)$ is a Cantor set $\iff$ $diam(P_n(x)) \to 0$ for every x.
- K_x is k-periodic $\iff$ the nest is k-periodic :

  $f^k(P_{n+k}(x)) = P_n(x)$ for $n \geq n_0$.

Branner-Hubbard Tableaux

The dynamics can be read on the diagonal of the tableaux

$P_0(x)$

$P_1(x)$

$P_2(x)$

$P_3(x)$

$\vdots$

$\vdots$

$P_n(x)$

$P_{n+1}(x)$

$\vdots$

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$$\begin{array}{ll}
P_0(x) & \\
 & f\nearrow \\
P_1(x) & \\
 & f\nearrow \\
P_2(x) & \\
 & f\nearrow \\
P_3(x) & \\
\vdots & \vdots \\
\vdots & \vdots \\
 & f\nearrow \\
P_n(x) & \\
 & f\nearrow \\
P_{n+1}(x) & \\
\vdots &
\end{array}$$

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$$
\begin{array}{ccc}
P_0(x) & & P_0(f(x)) \\
 & f\nearrow & \\
P_1(x) & & P_1(f(x)) \\
 & f\nearrow & \\
P_2(x) & & P_2(f(x)) \\
 & f\nearrow & \vdots \\
P_3(x) & & \vdots \\
\vdots & \vdots & \\
\vdots & \vdots & \\
 & & P_{n-1}(f(x)) \\
 & f\nearrow & \\
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\vdots & \vdots & & \\
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$$
\begin{array}{ccccc}
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 & f\nearrow & & f\nearrow & \\
P_1(x) & & P_1(f(x)) & & P_1(f^2(x)) \\
 & f\nearrow & & f\nearrow & \vdots \\
P_2(x) & & P_2(f(x)) & & \vdots \\
 & f\nearrow & \vdots & \vdots & \\
P_3(x) & & \vdots & \vdots & \\
\vdots & \vdots & & & P_{n-2}(f^2(x)) \\
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To prove that $K_x = \bigcap P_n(x)$ is reduced to $\{x\}$ one needs to understand this combinatorics and the following analysis.

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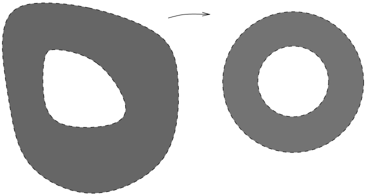

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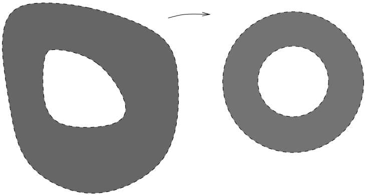

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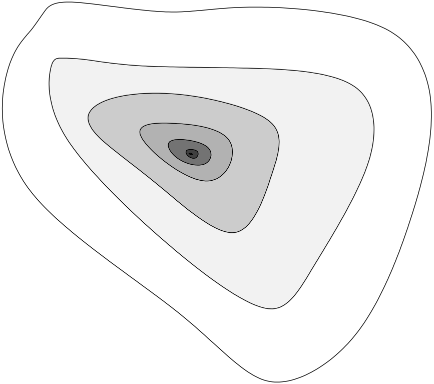
3. it is enough to prove that $\sum mod(A_n(x)) = \infty$.



Generally $f(A_{n+1}(x)) \neq A_n(f(x))$ for $A_n(x) = P_n(x) \setminus \overline{P_{n+1}(x)}$ but

- It is critical if $P_{n+1}(x)$ contains the critical point and $\bmod(A_n(x)) = \frac{1}{2} \bmod(A_{n-1}(f(x)))$.

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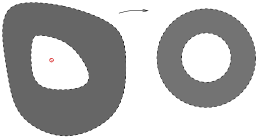

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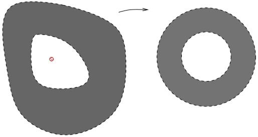

- It is semi-critical if $A_n(x)$ contains the critical point and $\mathrm{mod}\,(A_n(x)) \geq \frac{1}{2}\ \mathrm{mod}\,(A_{n-1}(f(x)))$.

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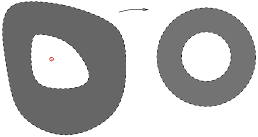

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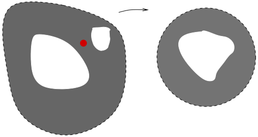

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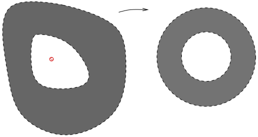

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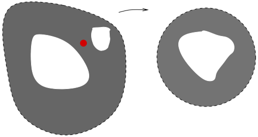


- It is non critical if $P_n(x)$ contains no critical point and $\mod(A_n(x)) = \mod(A_{n-1}(f(x)))$.

Two important properties :

- $\overline{P_{n+1}(x)} \subset P_n(x)$
- $P_n(x)$ is a topological disk

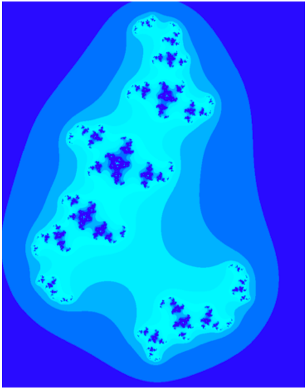
So that there is a non degenerate annulus $P_n(x) \setminus \overline{P_{n+1}(x)}$.

Remark : If K_x is l-periodic then

$$f^l : P_{n+l}(x) \to P_n(f^l(x)) = P_n(x)$$

is a covering of degree at most 2.

- If the degree is 2 then $f^l : P_{n+l}(x) \to P_n(f^l(x)) = P_n(x)$ is a polynomial like map of degree 2 so conjugate to some $z^2 + c$
- if the degree is 1 then $K_x = \{x\}$ is periodic.



Theorem (McMullen)

For a cubic polynomial f with Cantor Julia set, the Lebesque measure of $J(f)$ is zero.

If the puzzle pieces / graph does not separate the Julia set

then we just get $K_x = K(f)$

So the graph used has to cut the Julia set in two pieces at least.

Cut it properly i.e. in one point

Let $f : U \to V$ with $U \subset V$

Define a puzzle for the map f

by a finite connected graph $\Gamma \subset U$ satisfying

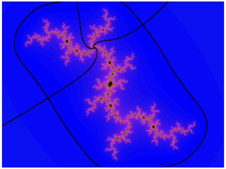
- $f(\Gamma) \cap U \subset U$
- the forward orbits of critical points are disjoint from Γ

The puzzle pieces of level n

are the connected components of $f^{-n}(U \setminus \Gamma)$

intersecting $K(f) = f^{-n}(U)$.

Yoccoz puzzles for quadratic polynomials



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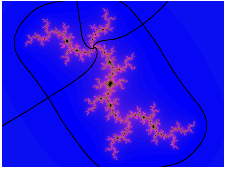 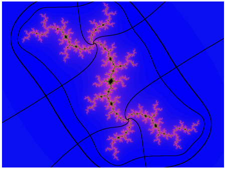

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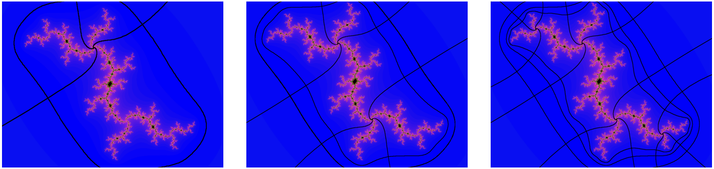

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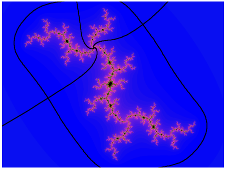

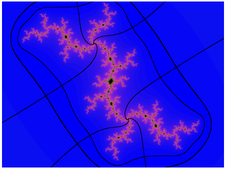

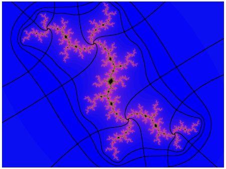

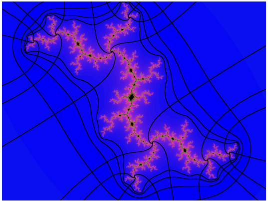

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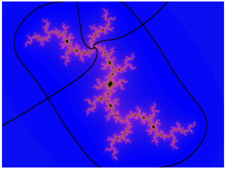
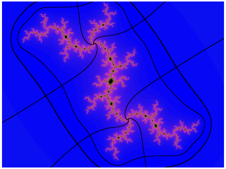
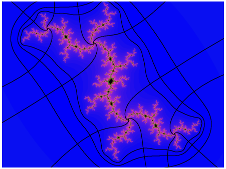
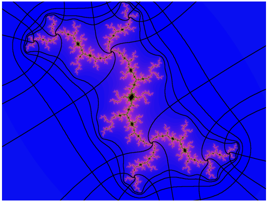
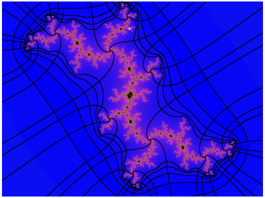






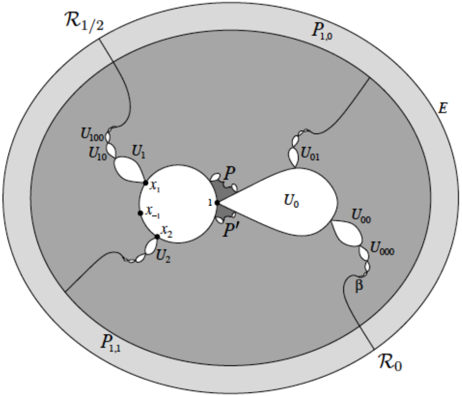




Yoccoz Theorem : The map is renormalizable or the impression of puzzle pieces is one point

Siegel disks

Carsten Petersen constructed a puzzle piece for Siegel disk working on the Blaschke model.



Petersen, Petersen-Zakeri : Most Siegel Julia sets are locally connected

Higher degree polynomials

Those constructions have been generalized for many cubic polynomials and in higher degree for polynomials . The goal is to prove that

- $\bigcap_{n\in\mathbf{N}} P_n(x) = \{x\}$
- or the map is renormalizable,
- then find another puzzle for the renormalized map.

There are several critical points and the degree is no more 2. New tools :

- develop the combinatorics by constructing particular nest called KSS-nest
- use analytic tools like Kahn-Lyubich covering Lemma.

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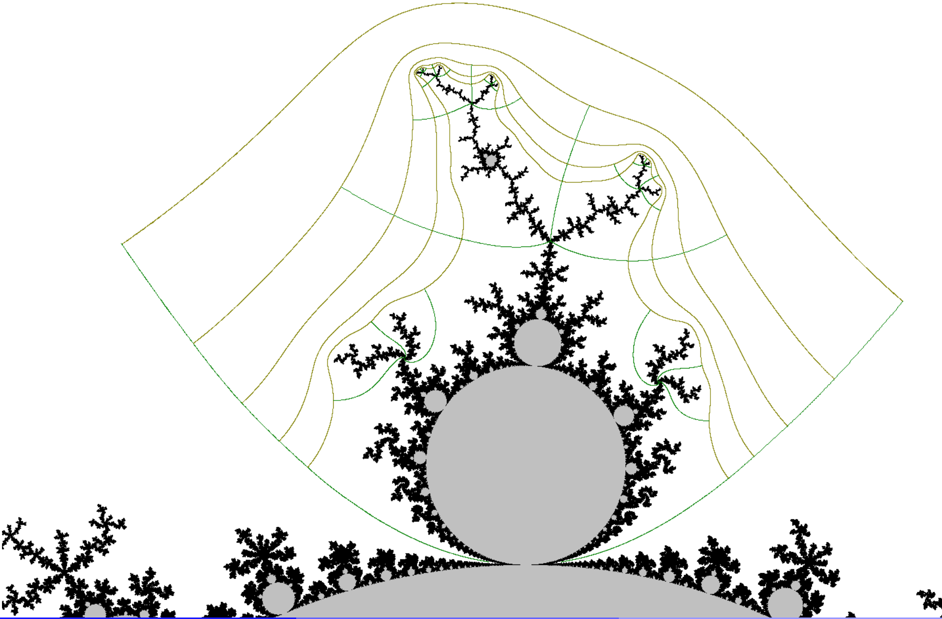
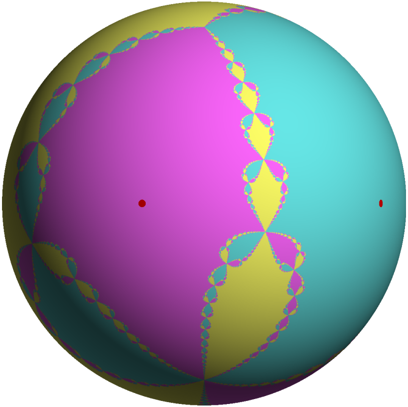
- one can get convergence of an access like external ray, since puzzle pieces can be used like prime-ends.
- one can get a description of a rational map as a mating using the conjugacy given by puzzles.
- one can get model in parameter space via puzzles in parameter spaces

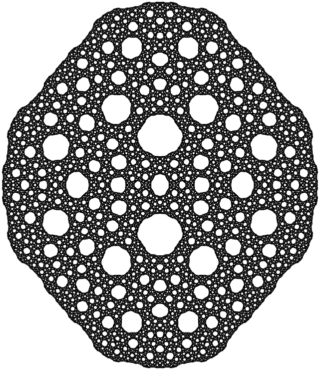


Rational maps

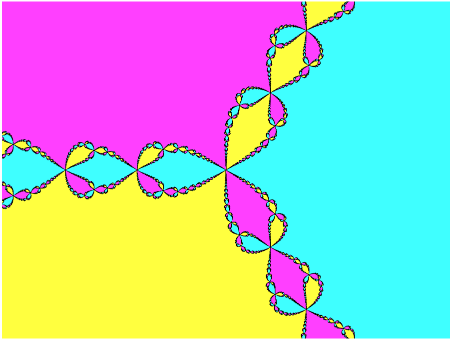
For rational maps there is no equipotential and rays cutting the Julia set like for polynomials

Julia set of a rational map is more complicate





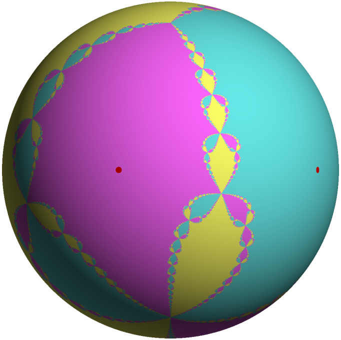
First example : cubic Newton map

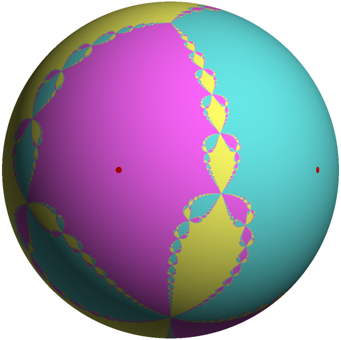
.

The *Newton's method* N_P of a polynomial P is defined by

$$N_P(z) = z - \frac{P(z)}{P'(z)}.$$

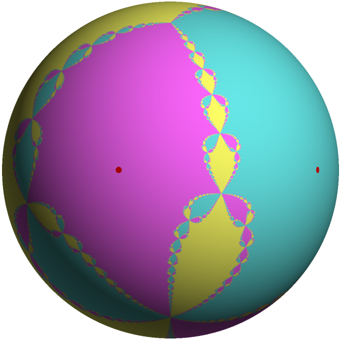
The roots of P are super-attracting fixed points of N_P.





The Julia set of a rational map is defined as the unique minimal compact subset of the Riemann sphere $\widehat{C}$ totally invariant (by N and N^{-1}) containing at least 3 points.

To cut the Julia set in small pieces we need to construct the equivalent to external ray.



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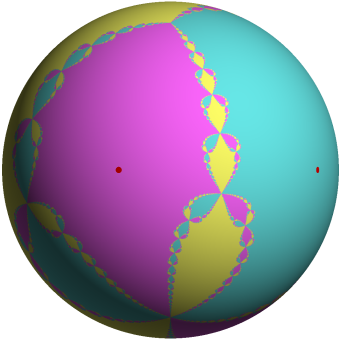
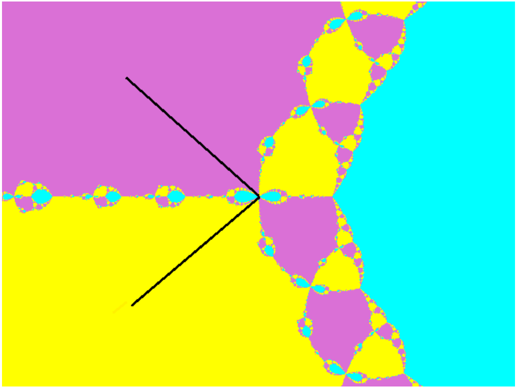


There are 3 basins corresponding to the 3 roots of P, ∞ is a common point, landing of fixed internal rays in the basins.

Except in the symmetric case, only two basins intersect and there is a last angle of intersection



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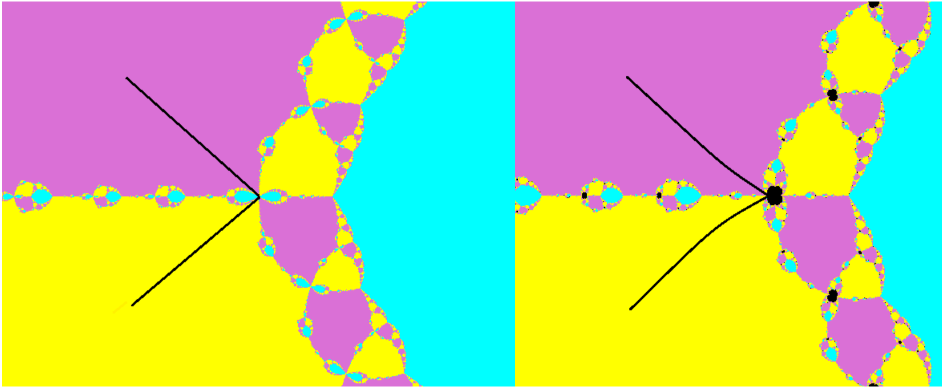
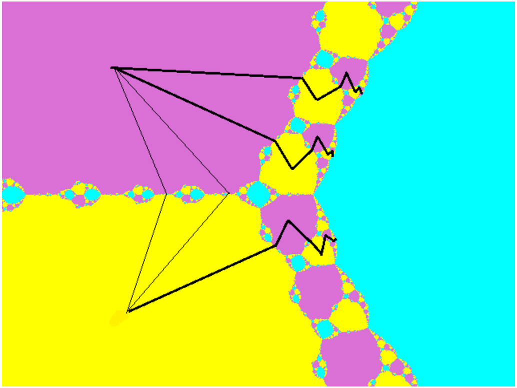


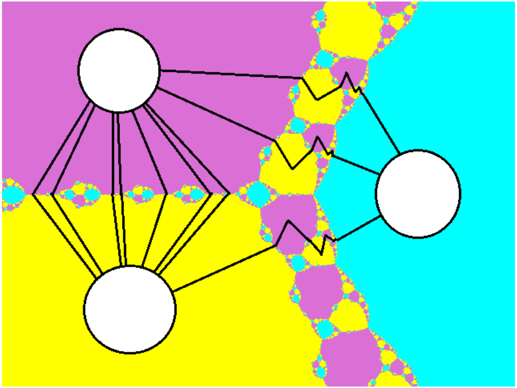
There is a Cantor set of angles Θ defining the intersection.

Construction of articulated rays by iterated pull back



It is a curve γ such that $f^k(\gamma) = \gamma \cup R_1(t) \cup \overline{R_2(-t)}$ with $t \in \Theta$. It consists in infinitely many internal rays alternating from basin 1 et 2.

Using the following two graphs,



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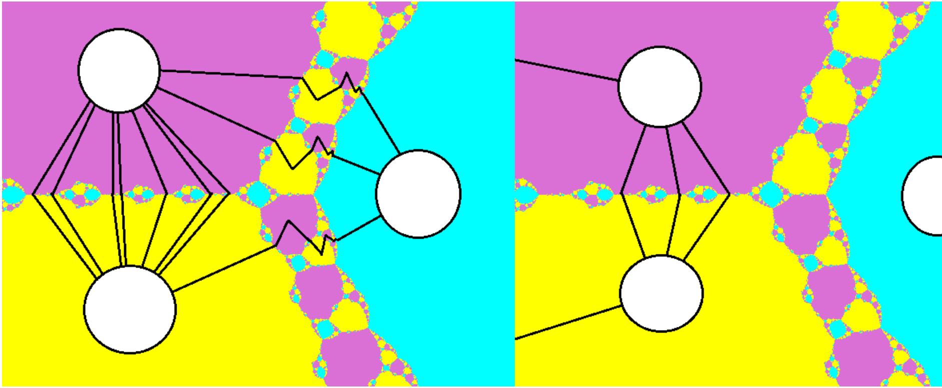
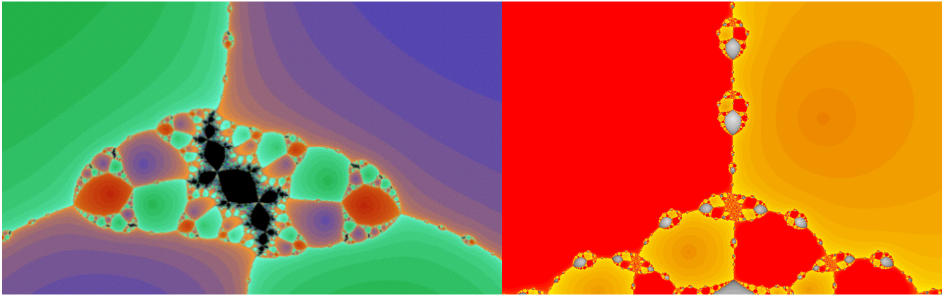


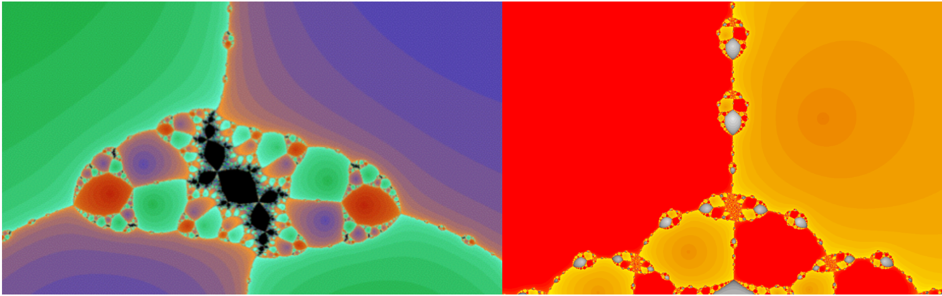
Theorem (R)

The intersection of the puzzle piece is either a point or the homeomorphic image of the filled Julia set of a quadratic polynomial.



Theorem (R)

In most cases the Julia set is locally connected.



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Theorem (R)

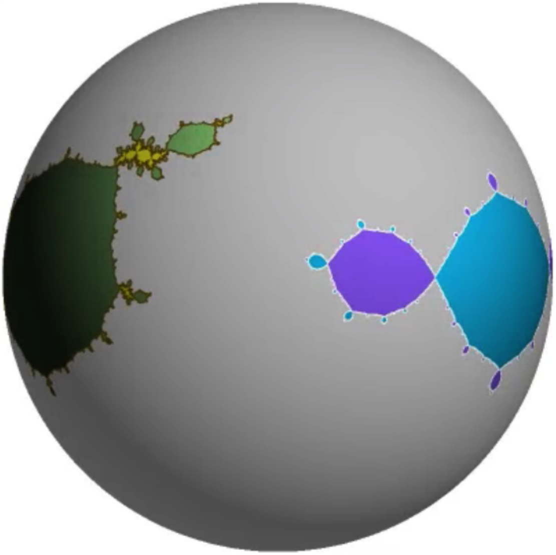
In particular $J(N) \supset h(J(P))$ where $J(P)$ is a non locally connected Julia set of quadratic polynomials P and $J(N)$ is locally connected.

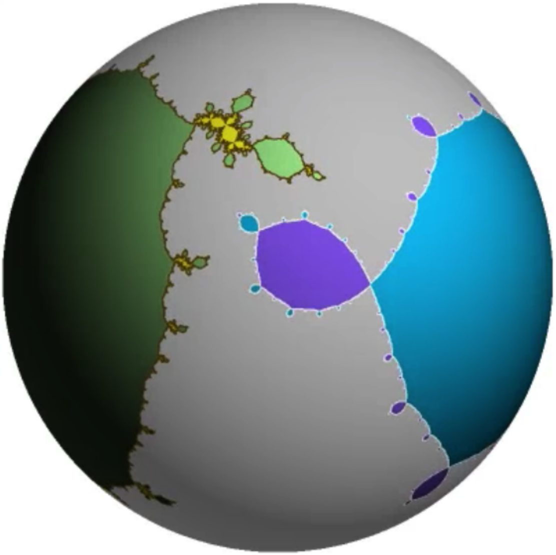
We use this puzzle structure to prove Tan Lei's conjecture

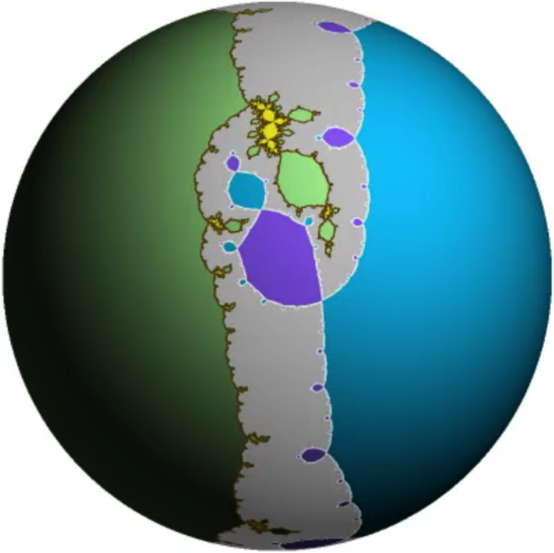
Theorem (Aspenberg, R)

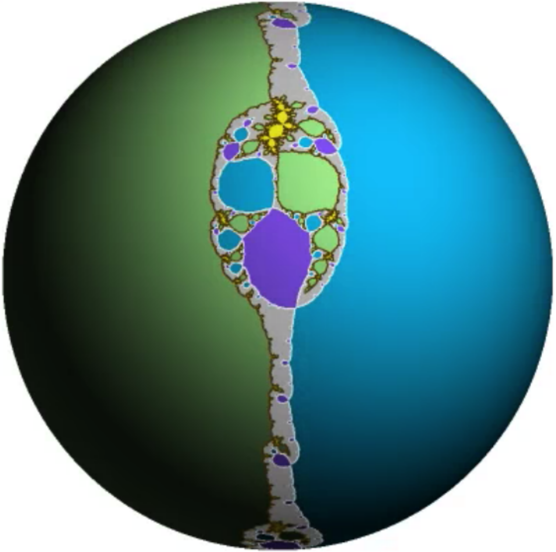
There exists a subset RC of renormalizable cubic polynomials, a subset RN of renormalizable cubic Newton maps and a map $M : RC \to RN$ which is onto and such that $M(f)$ is the mating of f with the polynomial $f_\infty(z) = z(z^2 + \frac{3}{2})$.

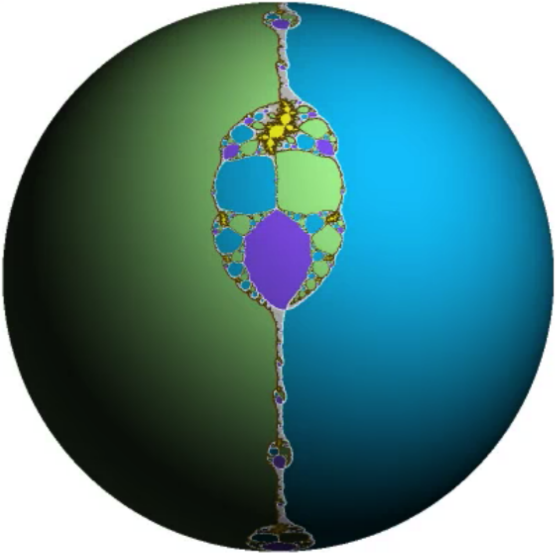
One can understand the dynamics of N through the dynamics of the polynomials. But there is no external rays any more.





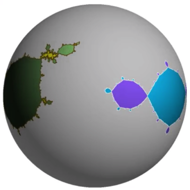






Sketch of the mating

Understand rational map via the two polynomials



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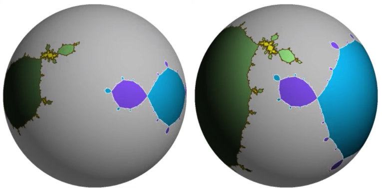

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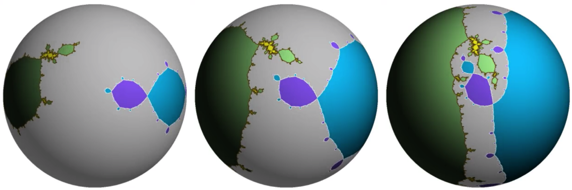

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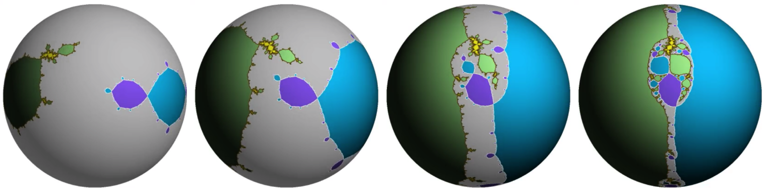

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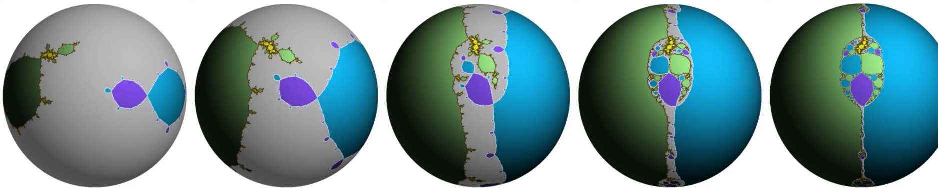
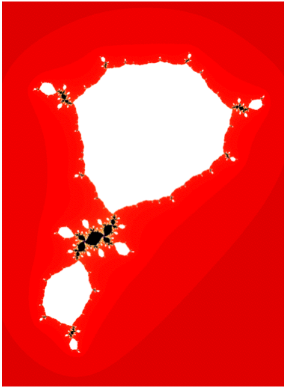


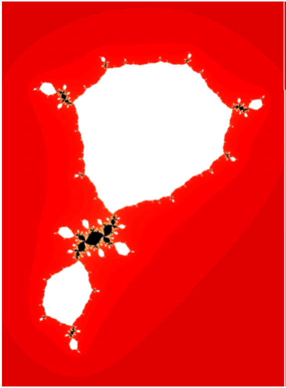
Definition

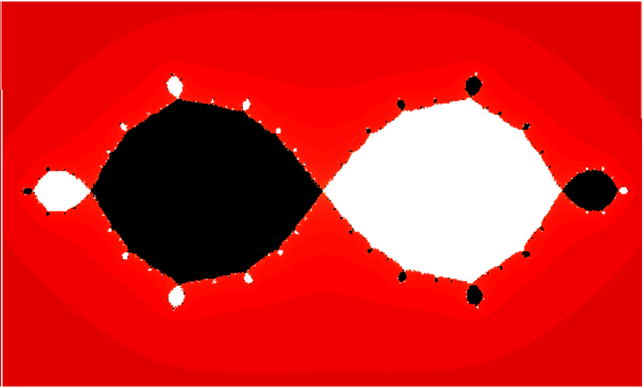
Two polynomials f_1 and f_2 are said mateable, if there exist a rational map R and two semi-conjugacies $\phi_j : K_j \to \hat{\mathbb{C}}$ conformal on the interior of K_j, such that $\phi_1(K_1) \cup \phi_2(K_2) = \hat{\mathbb{C}}$ and

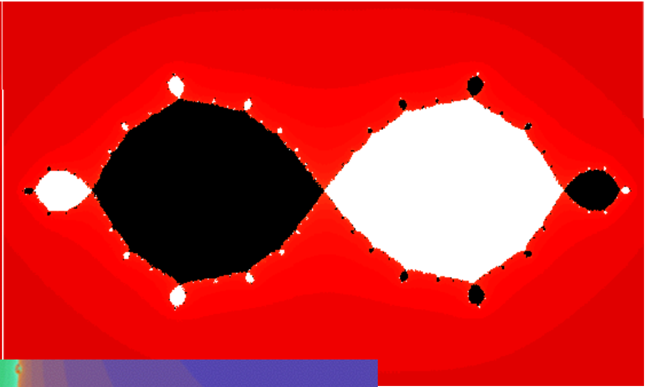
$$\forall (z, w) \in K_i \times K_j, \quad \phi_i(z) = \phi_j(w) \iff z \sim_r w.$$

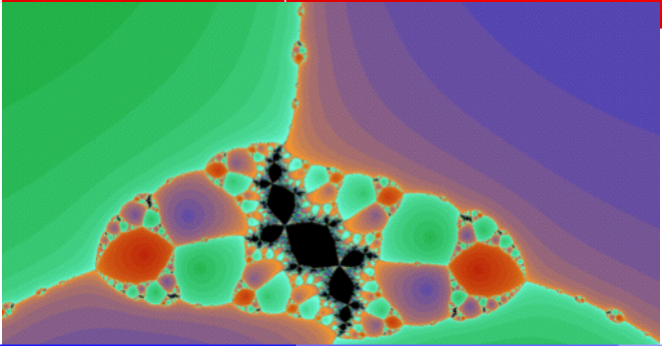
The relation $\sim_r$ is generated by :
the landing point of $R_1(t)$ is equivalent to the landing point of $R_2(-t)$.











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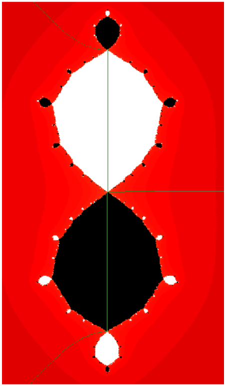
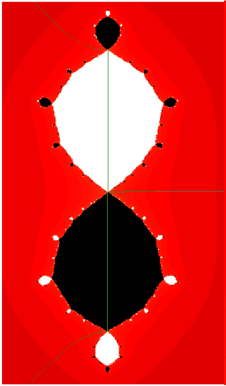
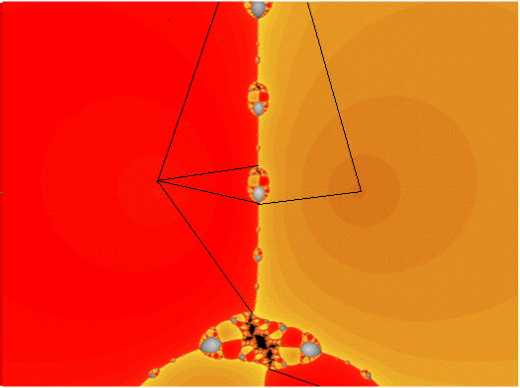
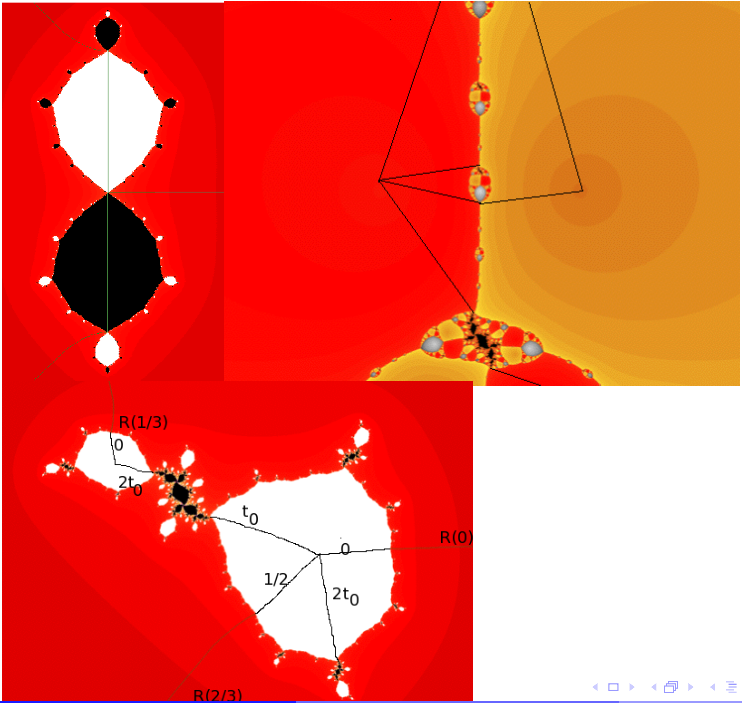
Idea of the proof : we construct the semi conjugacy by sending the puzzle pieces of the abstract mating to the puzzle pieces for the Newton map.

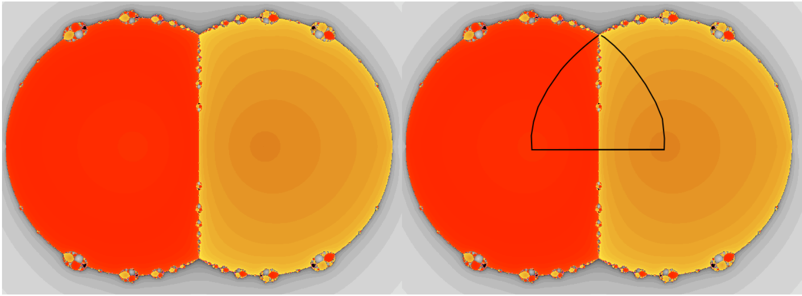







R(1/3)
0
2t0
t0
R(0)
0
1/2
2t0
R(2/3)

To find the cubic Newton map, one has to investigate the space of cubic Newton map.



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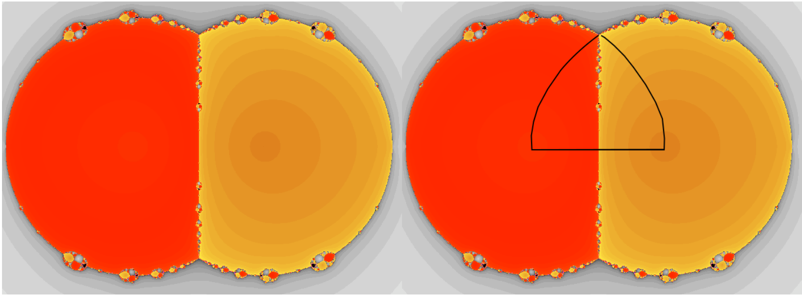

It is a one parameter slice with symmetries.
More precisely any Newton map is conjugate to one of the form

$$N_\lambda(z) = \frac{2z^3 - (\lambda^2 - \frac{1}{4})}{3z^2 - (\lambda^2 + \frac{3}{4})} \text{ with } \lambda \in \mathbb{C} \setminus \{\pm\frac{3}{2}, 0\}$$

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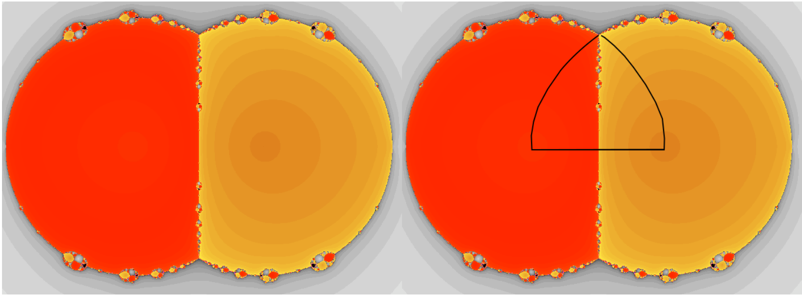

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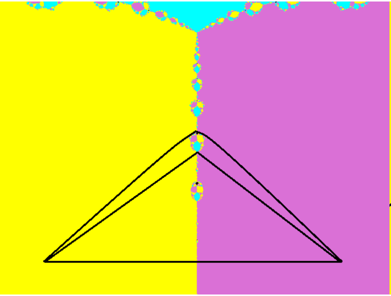
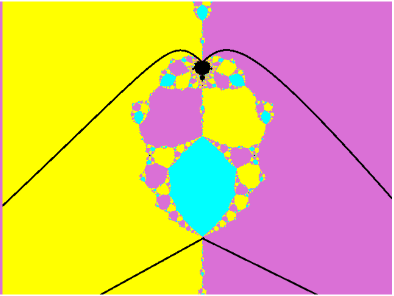
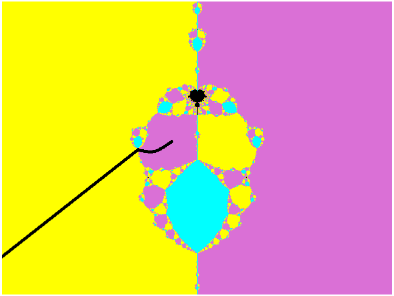
The graphs exist and define puzzles in some precise regions of the parameter plane called para-puzzle pieces.

To define them one has to transfer to the parameter plane the articulated



rays and all the pre-images.

Theorem (Wang, R, Yin)

(Advances 2017) Any ray in any hyperbolic component lands. The boundary of any hyperbolic component is a Jordan curve.
Rigidty : Two Newton maps with the same combinatorics are conformally conjugated.

It generalizes the proof done with para-puzzle pieces of the following

Theorem (R)

The boundary of the principal hyperbolic components are Jordan curves.

Sketch of the proof in the case of the principal hyperbolic component:

- Assume λ_1 and λ_2 are two accumulation points of an irrational ray so that $R_{\lambda_i}(t)$ lands at the free critical point of N_{λ_i}.
- Then the Newton maps N_{λ_1} and N_{λ_2} share the combinatorial dynamics with respect to the puzzles constructed with the same angles.
- There is a topological conjugacy ψ between N_{λ_1} and N_{λ_2}, which is holomorphic in the Fatou set of N_{λ_1}.
- Using control on the distortion, on the shape and the decreasing of puzzle pieces we get that the conjugacy is a quasi-conformal map.
- The Lebesgue measure of $J(N_{\lambda_i})$ is zero (Lyubich, Shishikura arguments on rational like maps with an admissible puzzle)
- The conjugacy is a Möbius transformation

More recent progress in the dynamical plane...

Theorem (Wang, Yin, Zeng)

Let f_p be the Newton map for any non-trivial polynomial P. Then the boundary of any immediate root basin B is locally connected.

This is proved by generalizing the work for cubic Newton maps. Namely the puzzles by applying KSS nest and Kahn Lyubich covering Lemma.

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Theorem (R., Yin, Zeng)

(arxiv. 11/2018) Non-renormalizable Newton maps are rigid. More precisely, we prove that the topological conjugacy is equivalent to quasiconformal conjugacy in this case.

Theorem (Drach, Lodge, Schleicher, Sowinski)

There exists an invariant graph for higher degree Newton maps that gives a Fatou-Shihikura injection.

Theorem (Drach,Schleicher)

Rigidity for non renormalisable Newton maps or in the same "way".

McMullen maps

We consider the maps

$$f_\lambda : z \mapsto z^n + \frac{\lambda}{z^n}.$$

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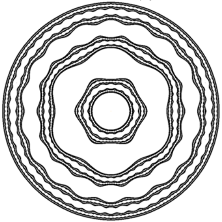
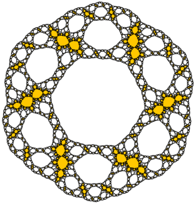
McMullen showed that the Julia set of f_λ is a Cantor set of simple closed curves provided
$n \neq 1, 2$ and λ is small.



We restrict to $n \geq 3$.

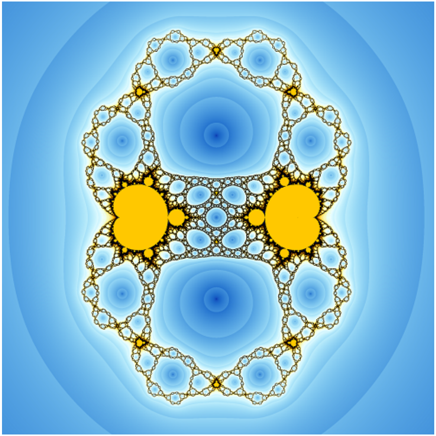
There exist also maps which are renormalizable and contain copies of



polynomial Julia sets.

In the parameter plane appear :

- the unbounded component which is the Cantor set region
- the neighborhood of 0 where $J(f_\lambda)$ is a Cantor set of circles
- the other "holes" where the Julia set is a Sierpinsky carpet.



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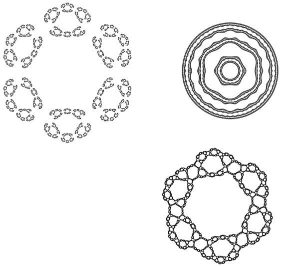

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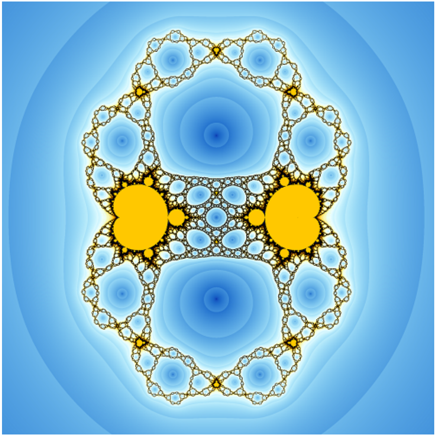

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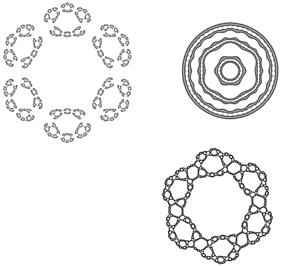
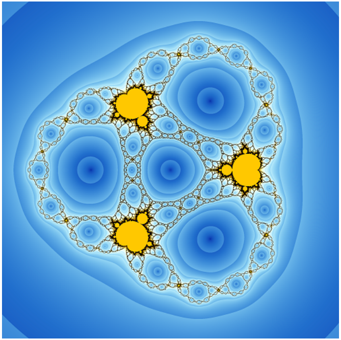
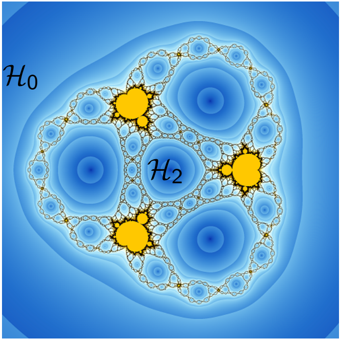


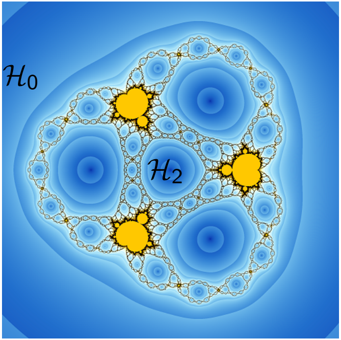
$\mathcal{H}_\infty$: the set of λ so that the critical points converge to ∞.





$\mathcal{H}_0$ is the unbounded component

$\mathcal{H}_2$ is the component contaning 0



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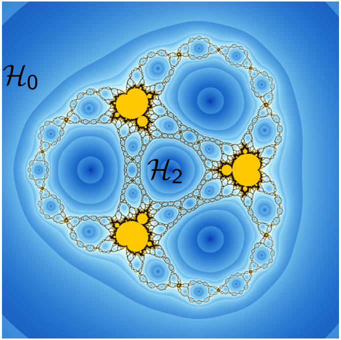

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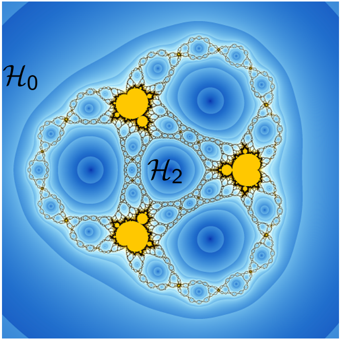
Precisely,

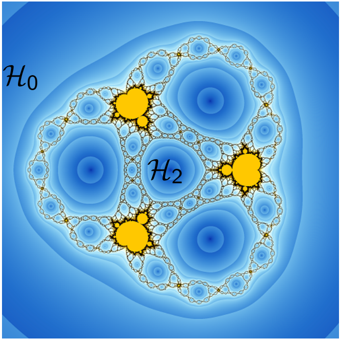
Theorem (Devaney-Look-Uminsky; Devaney-Russell)

- *If $\lambda \in \mathcal{H}_0$, then $J(f_\lambda)$ is a Cantor set ;*
- *If $\lambda \in \mathcal{H}_2 \setminus \{0\}$, then $J(f_\lambda)$ is homeomorphic to the product of a Cantor set and a circle ;*
- *If $\lambda \in \mathcal{H}_\infty \setminus (\mathcal{H}_0 \cup \mathcal{H}_2)$, then $J(f_\lambda)$ is a Sierpinsky carpet ;*
- *If $\lambda \notin \mathcal{H}_\infty$ then $J(f_\lambda)$ is connected.*



Theorem (Devaney)

The boundary of $\mathcal{H}_2$ is a Jordan curve.



Theorem (Devaney)

The boundary of $\mathcal{H}_2$ is a Jordan curve.

Conjecture (Devaney)

The boundary of any connected component of $\mathcal{H}_\infty$ is a Jordan curve.

Theorem (Qiu, Rœsch, Wang, Yin)

Let $\mathcal{H}$ be any connected component of $\mathcal{H}_\infty$. Then $\mathcal{H}$ is a Jordan domain.

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Proposition (Qiu, Rœsch, Wang, Yin)

The parametrization extends to the boundary as a function $\nu(\theta)$.

- If θ is periodic then the dynamical ray lands at a parabolic point.
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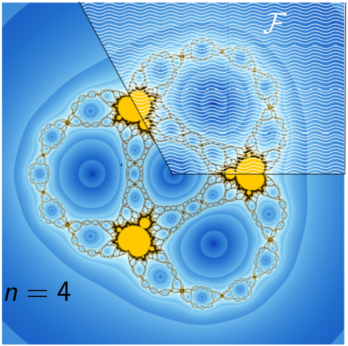
Corollary

The cusps are dense in the boundary of $\mathcal{H}_0$.

Some symmetries :

$$f_\lambda(\overline{z}) = \overline{f_{\overline{\lambda}}(z)} \quad \text{and} \quad f_\lambda(\omega z) = \omega f_{\lambda\omega^{-2}}(z) \quad \text{where} \quad \omega = e^{\frac{2i\pi}{n-1}}.$$

We will always restrict to the fundamental domain :



$$\mathcal{F} = \{\lambda \in \mathbf{C}^* \mid 0 \leq arg\lambda < \frac{2\pi}{n-1}\}$$

Some dynamics

The maps $f_\lambda(z) = z^n + \lambda/z^n$ are the composition of two simple maps

$$z \mapsto z + \frac{\lambda}{z} \quad \text{and} \quad z^n.$$

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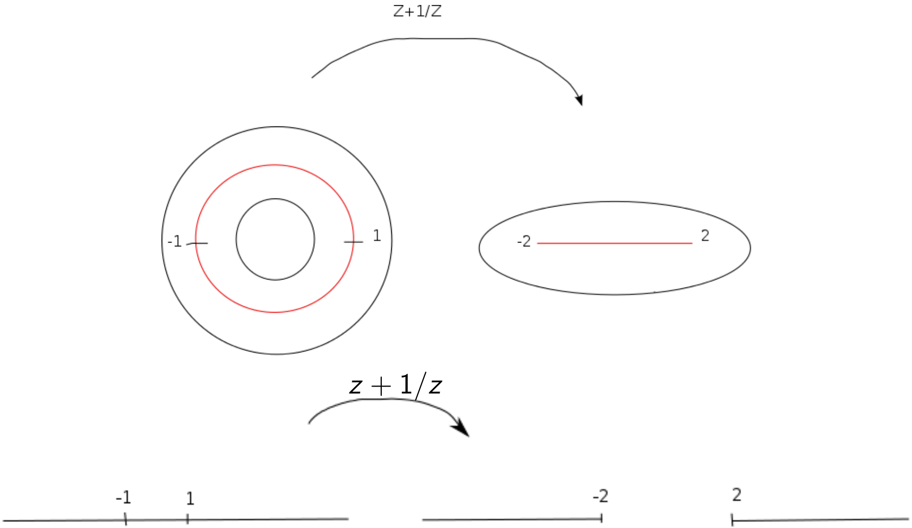
The map

$$z \mapsto z + \frac{\lambda}{z}$$

is just conjugated to

$$z \mapsto z + \frac{1}{z}.$$

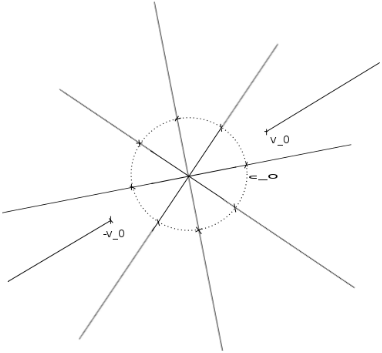

Z+1/Z
-1
1
-2
2
$z + 1/z$
-1
1
-2
2

The critical set of the map $f_\lambda(z) = z^n + \lambda/z^n$ is

$$Crit = \{0, \infty\} \cup C_\lambda$$

where

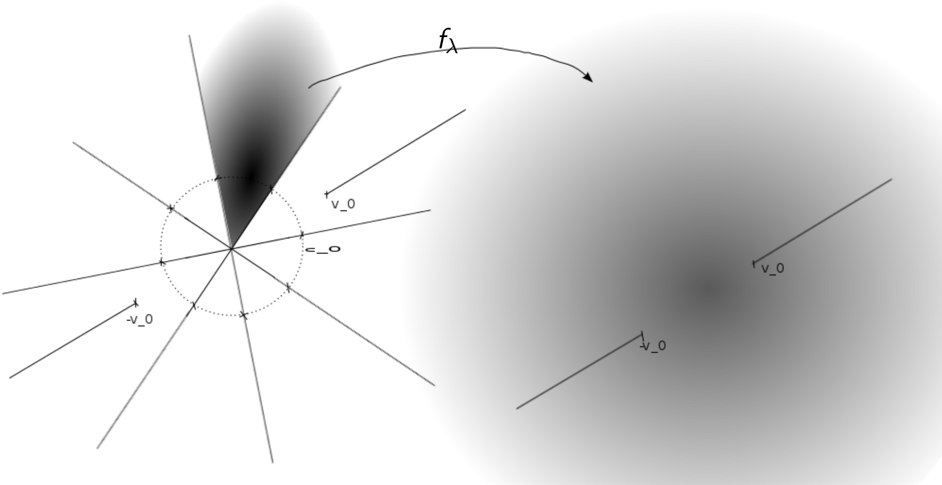
$$C_\lambda = \{c \mid c^{2n} = \lambda\} = \{c_0 e^{\frac{ik\pi}{n}} \mid k \in [0, .., 2n-1]\}$$



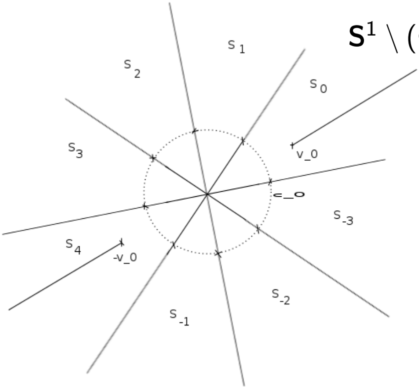
$n = 4$

.

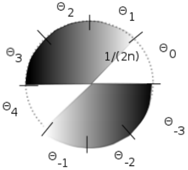
In each sector the map is one to one onto $\mathbf{C} \setminus \pm v_0[1, +\infty]$.



On can pull back any sector except the ones containing $\pm v_0$.



$$\mathbf{S}^1 \setminus (\Theta_0 \cup \Theta_n) = \left(\frac{1}{2n}, \frac{1}{2}\right] \cup \left(\frac{1}{2} + \frac{1}{2n}, 1\right]$$



$$\tau(\theta) = n\theta \mod 1.$$

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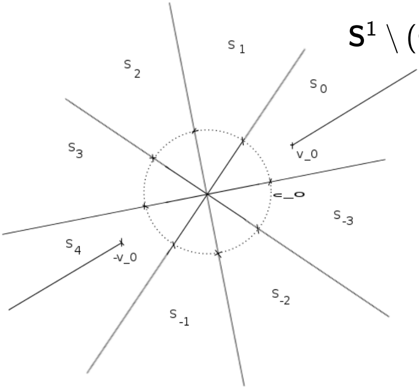

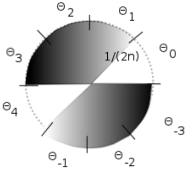

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θ has itinerary $(s_0, \cdots, s_k, \cdots)$

if $\tau^k(\theta) \in \Theta_{s_k}$

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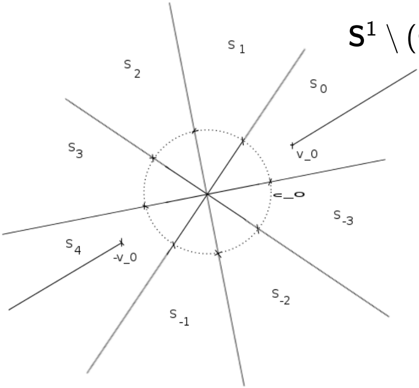

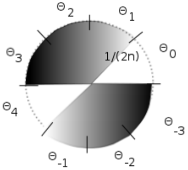

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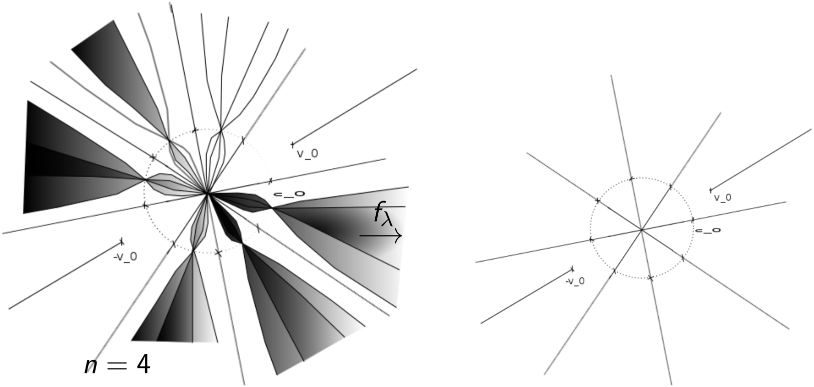
$$\Theta = \left\{\theta \mid \tau^k(\theta) \in \mathbf{S}^1 \setminus (\Theta_0 \cup \Theta_n) \quad \forall k \geq 0\right\}$$

Pulling back to the sectors without critical values



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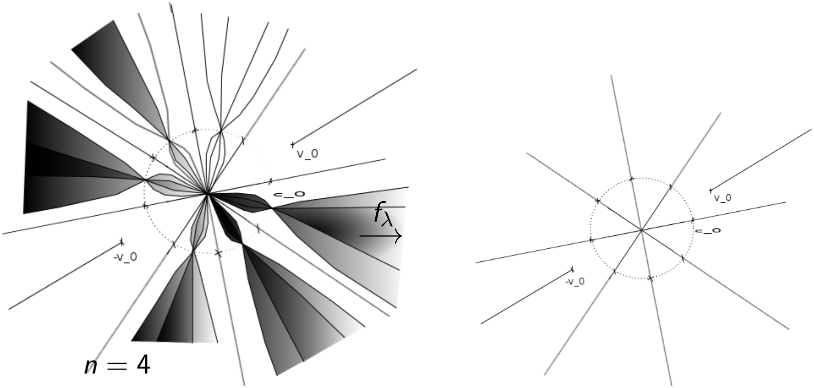


The intersection of a decreasing sequence of sectors shrinks to a curve in some cases.

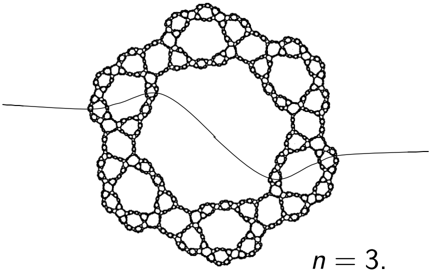
Theorem (Devaney, Qiu-Wang-Yin)

For any λ in the interior of the fundamental domain $\mathcal{F}$ and for any $\theta \in \Theta$ with itinerary $(s_0, s_1, \cdots,)$ the set

$$\Omega_\lambda^\theta := \bigcap_{k \geq 0} f_\lambda^{-k}(S_{s_k}^\lambda \cup S_{-s_k}^\lambda)$$

is a Jordan curve intersecting the Julia set under a Cantor set.

"cut rays"



$\Omega_\lambda^1 = \Omega_\lambda^{1/2}$

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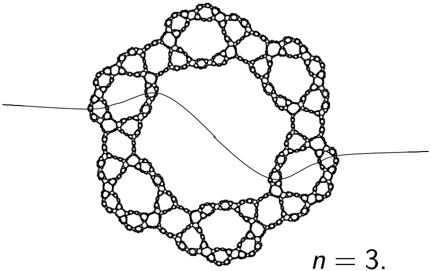

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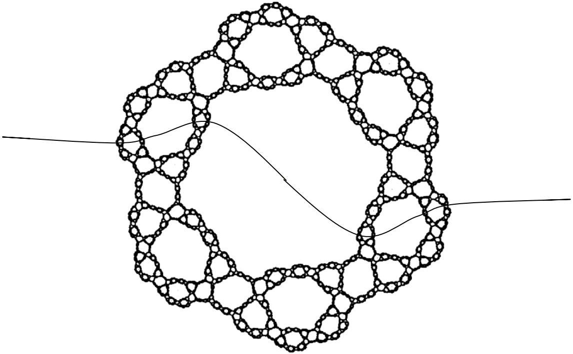
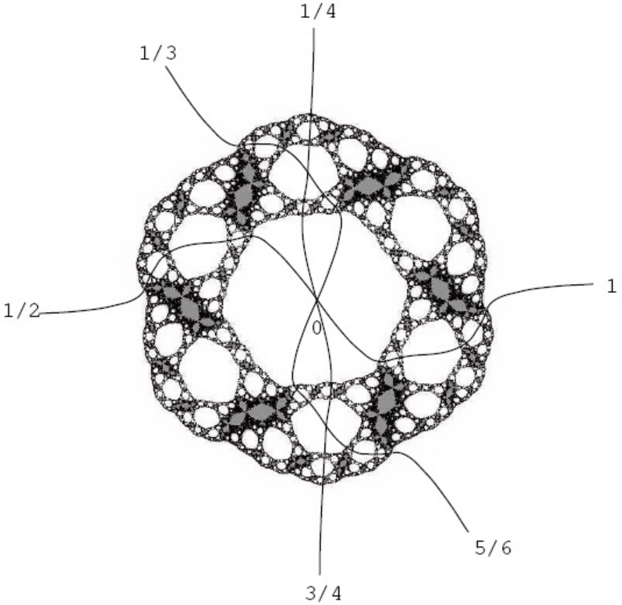
There is a similar construction for $\lambda \in \mathbf{R}$.




1/4
1/3
1/2
1
0
5/6
3/4

Notations :

- B_λ is the immediate basin of ∞,
- $\phi_\lambda : B_\lambda \to \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ the Böttcher map :

$$\phi_\lambda(f_\lambda(z)) = (\phi_\lambda(z))^n, \quad \phi'_\lambda(\infty) = 1,$$

- $R_\lambda(\theta)$ is the ray of angle θ in B_λ :

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Properties of "cut rays" :

- $\Omega_\lambda^\theta = -\Omega_\lambda^\theta = \Omega_\lambda^{\theta+1/2}$;
- $f_\lambda : \Omega_\lambda^\theta \to \Omega_\lambda^{\tau(\theta)}$ is two to one ;
- $\Omega_\lambda^\theta \cap B_\lambda = R_\lambda(\theta) \cup R_\lambda(\theta + 1/2) \cup \{\infty\}$;
- $\Omega_\lambda^\theta \cap (\overline{\mathbf{C}} \setminus J(f_\lambda)) \subset \bigcup_{k \geq 0} f_\lambda^{-k}(B_\lambda)$.

The "cut rays" are used in order to construct a puzzle.

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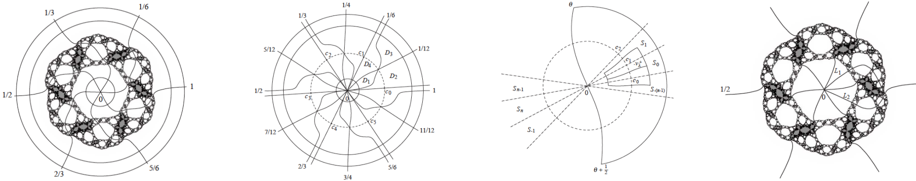


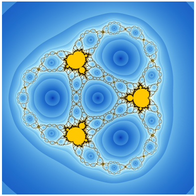
Theorem (Qiu-Wang-Yin)

If $J(f_\lambda)$ is not a Cantor set, then the boundary of B_λ is a Jordan curve.

The precise result is on decreasing of puzzle pieces , it is used in order to get the rigidity in the parameter plane.

Parameter plane.

We restrict to $\mathcal{H}_0$.



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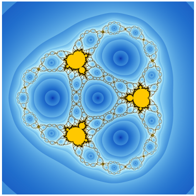

To prove that $\partial\mathcal{H}_0$ is a Jordan curve we will prove that the impression of any ray is reduced to a single point.

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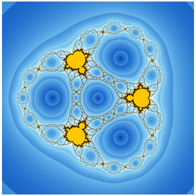


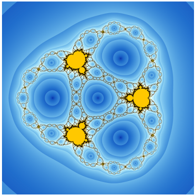
To prove that $\partial\mathcal{H}_0$ is a Jordan curve we will prove that the impression of any ray is reduced to a single point.

Let $\Phi_0 : \mathcal{H}_0 \to \overline{\mathbf{C}} \setminus \overline{\mathbf{D}}$ be a parametrization given by " the position of critical value in B_λ" :

$$\Phi_0(\lambda) = (\phi_\lambda(v_\lambda))^2.$$

Parameter plane.

We restrict to $\mathcal{H}_0$.



To prove that $\partial\mathcal{H}_0$ is a Jordan curve we will prove that the impression of any ray is reduced to a single point.

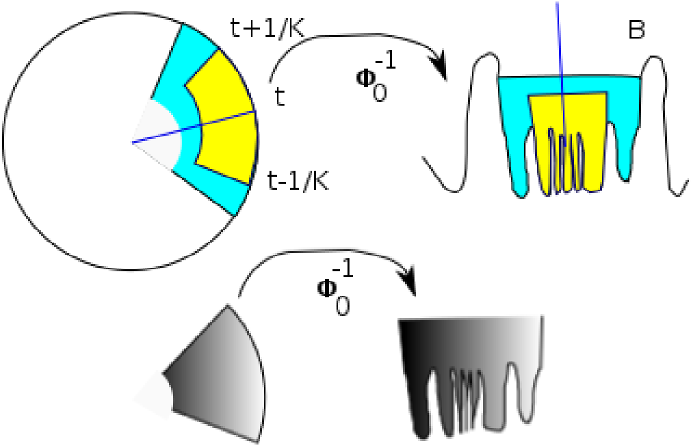
Let $\Phi_0 : \mathcal{H}_0 \to \overline{\mathbf{C}} \setminus \overline{\mathbf{D}}$ be a parametrization given by " the position of critical value in B_λ" :

$$\Phi_0(\lambda) = (\phi_\lambda(v_\lambda))^2.$$

A ray $\mathcal{R}(t)$ is $\Phi_0^{-1}(]1, \infty]e^{2i\pi t})$.

The *impression of a ray* is the intersection :

$$Imp(t) := \bigcap_{k\geq 1} \overline{\Phi_0^{-1}(\{]1, 1+1/k[e^{2i\pi\theta} \mid |\theta - t| < 1/k\})}$$



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3. For two parameters in the same impression, which are not cusps, we construct a conjugacy between the maps and use
   - Thurston's theorem in the post-critically finite case ;
   - that the homeomorphism is quasi-conformal, conformal on the Fatou set and that the Julia set is of measure zero.

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Therefore there exist U_λ, V_λ disks with $\overline{B_\lambda} \subset V_\lambda \subset \overline{V_\lambda} \subset U_\lambda$ such that $f_\lambda : V_\lambda \to U_\lambda$ is a polynomial-like map with one critical point.

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For nearby λ, one should have a polynomial-like map with the same degree. In $\mathcal{H}_0$, $J(f_\lambda)$ is a Cantor set.

Contradiction

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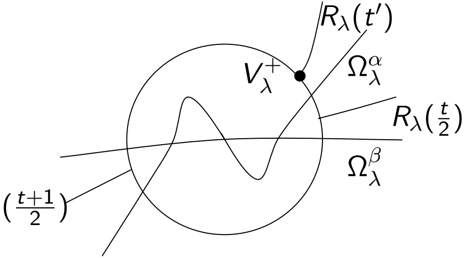

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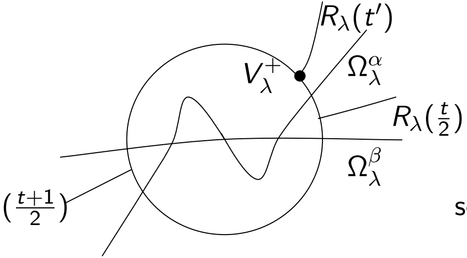

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Lemma

If $0 \leq t < 1/(n-1)$ and $\lambda_1, \lambda_2 \in Imp(t)$ are not cusps, then $R_{\lambda_1}(t/2)$ lands at v_{λ_1} and $R_{\lambda_2}(t/2)$ lands at v_{λ_2}.

By continuity.

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- then ψ is a Möbius conjugacy between f_{λ_1} and f_{λ_2}.

Rigidity for non-recurrent exponential maps

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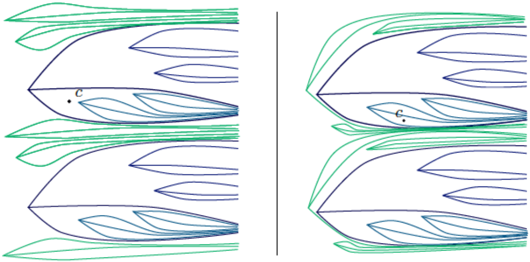
For each n, the the connected components of $\mathbf{C}\setminus$ where $\Gamma_n = \cup_{j=0}^{n} f^{-j}(\Gamma)$ are puzzle pieces of level n.



A parameter c is combinatorially non-recurrent if there is a suitable Γ which separates the singular value from the postsingular set.

Theorem[Benini]

Let $c; c_0$ be non-escaping parameters, and f_c be combinatorially non-recurrent. If f_{c_0} is combinatorially equivalent to f_c, then $c_0 = c$.