Complex dynamics: the intriguing case of wandering domains

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Joint work with Anna Miriam Benini, Vasiliki Evdoridou, Nuria Fagella and Phil Rippon

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The **Julia set** (or chaotic set) is

$$J(f) = \mathbb{C} \setminus F(f).$$

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Periodic Fatou components are well understood and there is a classification essentially due to Fatou and Cremer (1920s).

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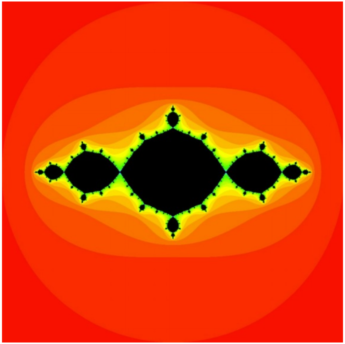

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Classification of invariant Fatou components

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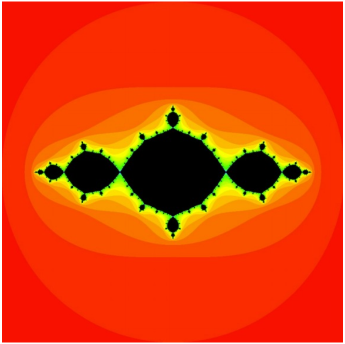

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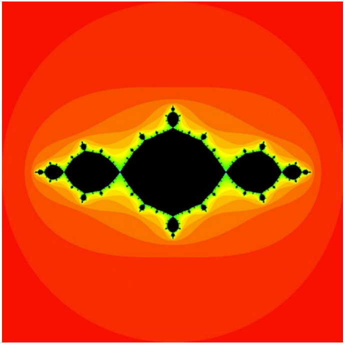

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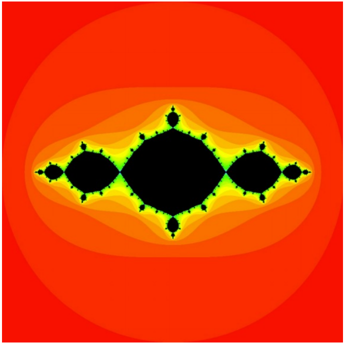

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- U is super-attracting if $f'(z_0) = 0$

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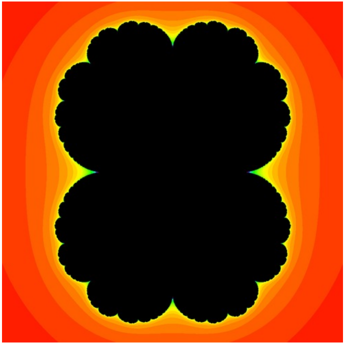

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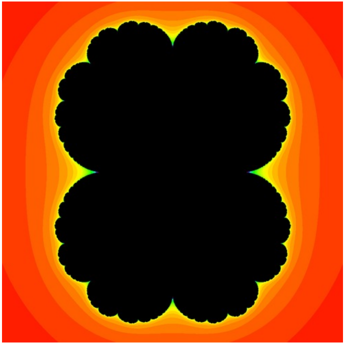

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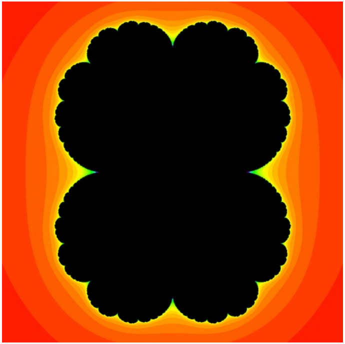

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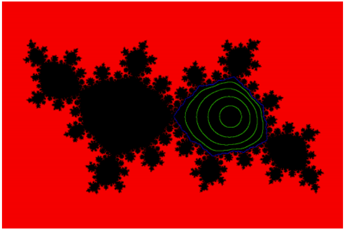

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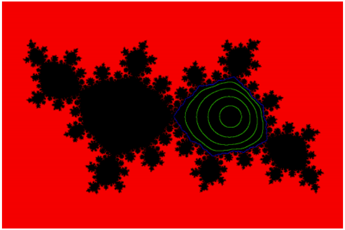

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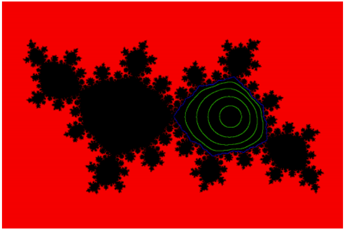

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- $f : U \to U$ is conjugate to an irrational rotation of the disc

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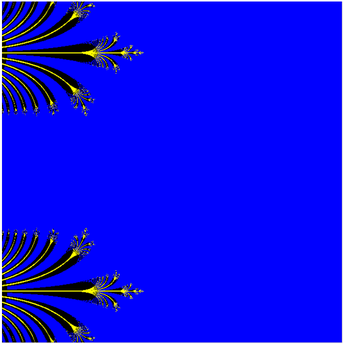

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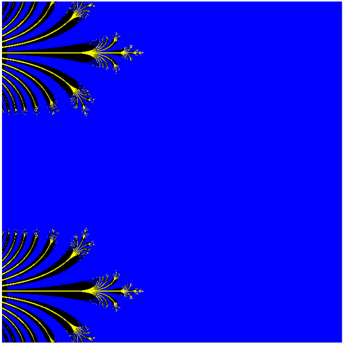

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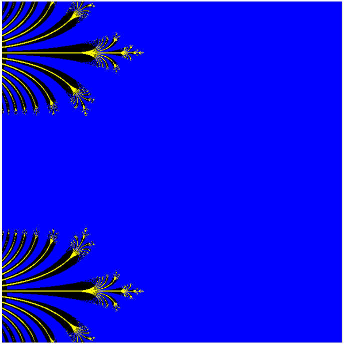

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- This type cannot occur for polynomials

The existence of wandering domains

Theorem (Sullivan, 1982)

If f is rational, then f has no wandering domains.

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Wandering domains *do* exist for transcendental entire functions, and are not well understood.

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  - In 1963 he constructed an infinite product f and a nested sequence of annuli A_n tending to infinity with $f(A_n) \subset A_{n+1}$.
  - In 1976 he showed that these annuli belong to distinct Fatou components (multiply connected wandering domains).

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Theorem (Baker, 1984)

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If U is a multiply connected wandering domain then there exist sequences (r_n) and (R_n) such that, for large n,

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$$R_n/r_n \to \infty \quad \text{as } n \to \infty.$$

Dynamical behaviour in multiply connected wandering domains

Theorem (Bergweiler, Rippon and Stallard, 2013)

If U is a multiply connected wandering domain then

- *for large $n \in \mathbb{N}$, there is an absorbing annulus*

$$B_n = A(r_n^{a_n}, r_n^{b_n}) \subset U_n$$

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This led to progress on a longstanding question as to whether $f \circ g = g \circ f$ implies $J(f) = J(g)$ (Benini, Rippon and Stallard).

Classifying simply connected wandering domains

EPSRC funded project

Gwyneth Stallard

Phil Rippon

Vasso Evdoridou

Nuria Fagella

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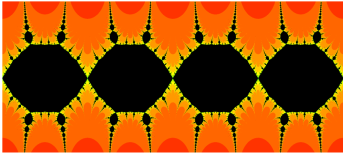

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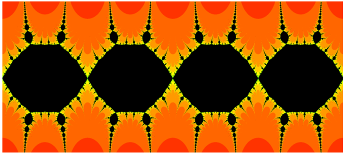

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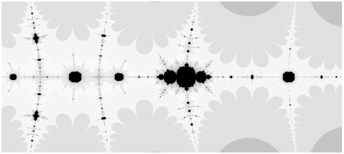

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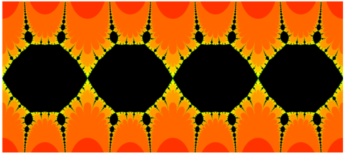

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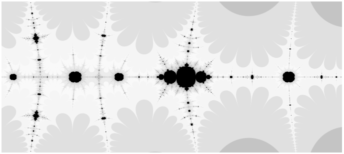

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Answer No - everything seems possible!

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- Show that the rate of contraction depends on the values of $g_n'(0)$ - using techniques of Beardon and Carne.

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- Show that the rate of contraction depends on the values of $g_n'(0)$ - using techniques of Beardon and Carne.

$$\sum_{n=0}^{\infty}(1 - |g_n'(0)|) = \infty \iff g_n(w) \to 0 \text{ as } n \to \infty$$

for all $w \in \mathbb{D}$.

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Classifying simply connected wandering domains

Distance from boundary

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- Vasso will tell you about this tomorrow!