From Polynomial to Rational Maps: Newton's Method as a Dynamical System

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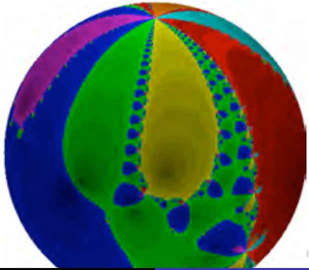
Barcelona, 26 March 2019



2 Overview

Guiding principle of talk: *Holomorphic dynamics has accumulated a lot of deep knowledge especially on polynomials. Rational maps seem much harder.*
Principle: *rational dynamics is no more difficult than polynomial dynamics once we have a good combinatorial structure.*

Report on multi-year project on dynamics of Newton dynamics; outline of different ingredients and key difficulties.

1. Quadratic polynomials, local connectivity: topological models, different orbits. Douady/Hubbard, Yoccoz et al
2. Quadratic parameter space, MLC: topological model, different dynamics
3. Newton maps as dynamical systems, cubic case
4. The fundamental ingredients in building up the theory
5. Newton puzzles and the Fatou–Shishikura-injection
6. Thurston theory for Newton maps
7. Trivial fibers for Newton maps and rational rigidity

3 Dynamics of quadratic polynomials

Introduction: 1980's, Douady/Hubbard, Yoccoz, Thurston

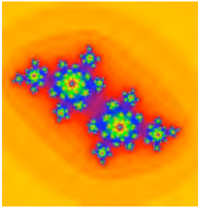
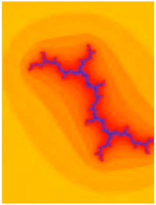
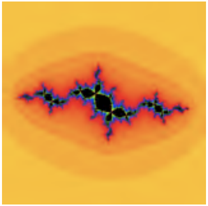
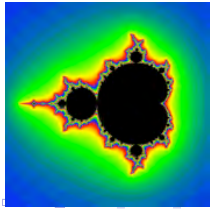
Iterate $p_c(z) = z^2 + c$ on $\mathbb{C}$. Interesting set:
Filled-in Julia set: $K_c := \{z \in \mathbb{C} \colon z \text{ has bounded orbit}\}$.

Decisive: orbit of critical value c.

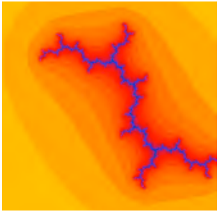
a) Critical orbit unbounded: K_c is Cantor set, dynamics on K_p is shift on sequences over 2 symbols. All dynamics "same".
$\Longrightarrow$ boring!

b) Critical orbit bounded ($c \in K_c$): then K_c is connected, topologically very interesting!

Define $\mathcal{M} := \{c \in \mathbb{C} \colon c \in K_c\}$: the **Mandelbrot set.**

4 (Some of) the relevant questions



Julia sets are complicated — describe:
a) simple models for the topology
b) simple models for the dynamics on the Julia set
c) are all orbits in the Julia set different? Can they be combinatorially distinguished?

Two relevant concepts:

• Julia set is **locally connected** (every point has arbitrarily small connected neighborhoods)
• every point z has **trivial fiber** (dynamics of z can be distinguished from all other orbits).

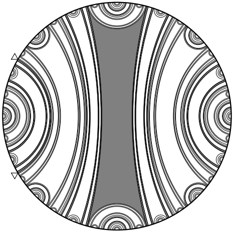
Observation: both concepts are equivalent (in most cases).

Theorem (Douady/Hubbard/Yoccoz/Lyubich/... 1980–1995
"Most" quadratic Julia sets are locally connected.

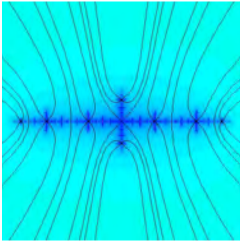
Theorem (Thurston/Douady) Have nice topological models for Julia sets in terms of *invariant laminations* and *pinched disks*.

5 Invariant laminations and pinched disks

Thurston: an invariant quadratic lamination is determined by a single angle (Thurston)

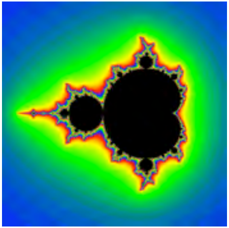


Douady: the topology of the Julia set is described completely by the "pinched disk" of the lamination: take the quotient of $\overline{\mathbb{D}}$, collape all leaves
The dynamics on the Julia set is the quotient of *angle doubling on the unit circle*.



This works for *all* quadratic Julia sets, unless they are **renormalizable** (small embedded polynomial dynamics) or have irrationally indifferent periodic points (and often in these cases too).

6 Parameter space: the Mandelbrot set $\mathcal{M}$



Define the **Mandelbrot set** $\mathcal{M} := \{c \in \mathbb{C}:$ the Julia set K_c is connected$\}$.

Every $c \in \mathcal{M}$ describes its own dynamical system $p_c(z) = z^2 + c$.

Relevant questions:

a) find a simple model for the topology of $\mathcal{M}$

b) are all dynamical systems p_c different (for $c \in \mathcal{M}$)?
(They are topologically the same for $c \notin \mathcal{M} \Longrightarrow$ boring!).

Two analogous relevant concepts in parameter space:

• the **Mandelbrot set is locally connected** $\Longrightarrow$ simple topology model

• every $c \in \mathcal{M}$ has **trivial fiber**: the dynamics of p_c can be combinatorially distinguished from all other orbits.

Again: both questions equivalent!

Analogous theory of “quadratic minor lamination” for $\mathcal{M}$.

7 From Newton dynamics to rational rigidity

Project goal: carry over successful theory of dynamics on quadratic polynomials to large family of rational Newton maps of all degrees.

- *parameter space:* distinction and classification of different (postcritically finite and beyond) Newton maps
- *dynamics of Newton maps:* description of Julia sets; all fibers are trivial or renormalizable
- ... and Newton is great as a *root finder*! (Not today.)

Moral of the story: rational dynamics is not harder than polynomial dynamics: the dynamics is *easy* unless polynomial dynamics interferes! All difficulties can be boxed and sent to the "department of polynomials". (At least in case of Newton maps.)

Rational Rigidity Principle (for Newton Maps)

8 Cubic Newton maps

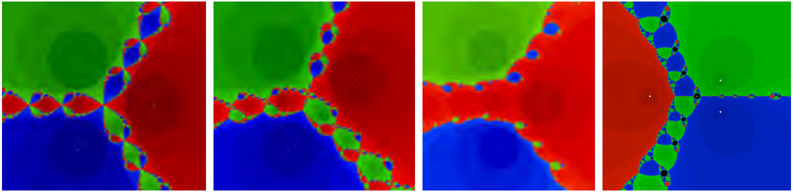
Goal of research on quadratic polynomials: “should pave the way for general holomorphic dynamics on $\mathbb{P}$.

One example where that works: Newton maps of cubic polynomials.

Polynomial $p(z) = c(z - a_1)(z - a_2)(z - a_3)$,
$N_p(z) = z - p(z)/p'(z)$

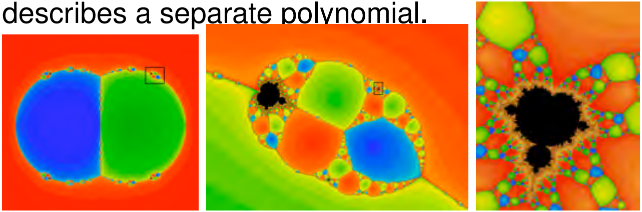
Convenient coordinates: $a_1 = 0$, $a_2 = 1$; factor c cancels in N_p

Hence $p(z) = z(z - 1)(z - \lambda)$: one complex parameter



Smale’s observation: there are cubic Newton maps with open disks of non-convergence! (Attracting cycles.)

Special case: cubic polynomials $p_\lambda(z) = z(z-1)(z-\lambda)$ (classical) Complex parameter space (λ-plane); every $\lambda \in \mathbb{C}$ describes a separate polynomial.

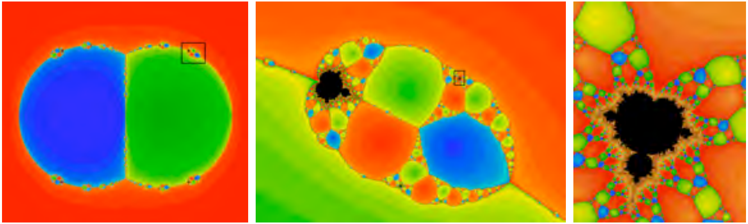


Black: parameters λ for which p_λ has an attracting cycle. Classified by countable collection of “little Mandelbrot sets”.

Colors: dynamics of the “free critical point” $c = (0+1+\lambda)/3$ (determines the global dynamics).

Theorem: *Every colored component has a unique center forwhich the free critical point has finite orbit (preperiodic). Most (all [*]) components of the little Mandelbrot sets have unique centers for which the free critical point has finite orbit (periodic).*

[*] if all fibers of Mandelbrot set trivial

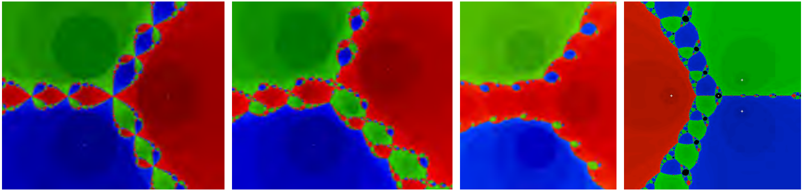


Theorem: *Every colored component has a unique center forwhich the free critical point has finite orbit (preperiodic). Most (all[*]) components of the little Mandelbrot sets have unique centers for which the free critical point has finite orbit (periodic).*

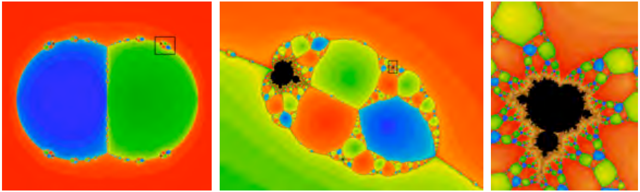
Classification (Tan Lei, Roesch, Wang, Yin, since 1990's) of cubic Newton dynamics in terms of:
a) hyperbolic components (colored), through their centers (in which dynamics is "postcritically finite"
b) little Mandelbrot sets (renormalizable dynamics)

We now understand cubic Newton map as well as the Mandel-brot set.

11 Different Newton dynamical systems



Top row: different cubic Newton dynamical systems



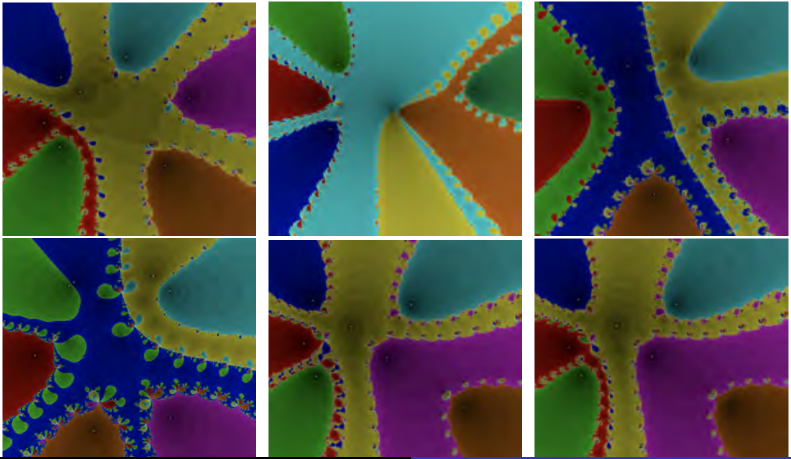
Bottom row: the cubic Newton parameter space (λ-plane)

Different “hyperbolic components” in parameter space (bottom) correspond to different positions of “free” the critical point in the Newton dynamics (top). At the *component center*, the free critical point lands on root.

12 Distinguish different Newton dynamics

Move from one degree of freedom (cubics) to general case!

Example: several rational maps of degree 7: Newton maps of degree 7 polynomials. Colors distinguish basins of different roots. Basin components can be connected in different ways.

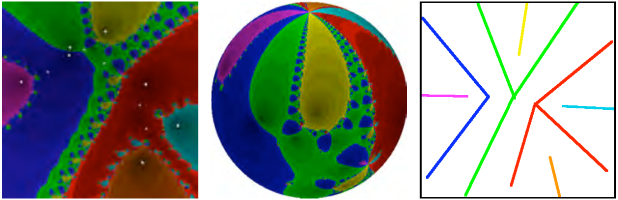


13 Newton dynamics in general: the beginning of the theory

Build theory of Newton dynamics in analogy to polynomial dynamics

Step 1 (Przytycki, 1989): Every immediate basin is simply connected, hence a Riemann domain

Step 2 (routine): change dynamics so that in every immediate basin, all critical points coincide ("attracting-critically-finite"; surgery)

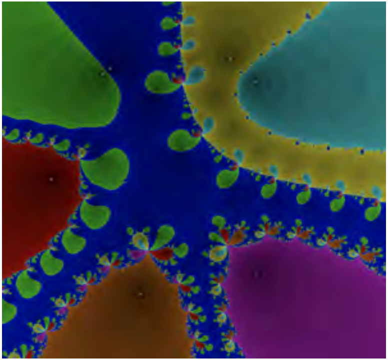


Step 3 (Hubbard-S.-Sutherland 2001): accesses to ∞ in immediate basins yield *channel diagram*: first step towards combinatorics

14 First hard step: connect the bubbles

Kostiantyn Drach, Yauhen Mikulich, Johannes Rückert, S.:

A combinatorial classification of postcritically fixed Newton maps; Ergodic Theory & Dynamical Systems 2019



a) there is a finite preimage of the channel diagram that contains all poles

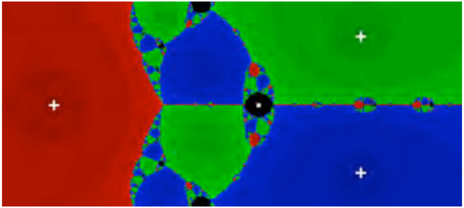
b) every “bubble” can be connected to an immediate basin via a finite chain of bubbles

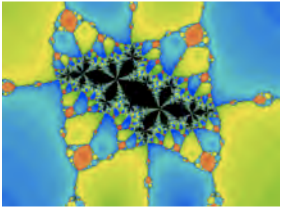
c) any two bubbles can be connected to each other through finitely many bubbles within $\mathbb{C}$

This provides global coordinate system for *all* Newton maps

Specifically in the *postcritically finite* case when all critical orbits are contained in (closures) of bubbles, we obtain a complete classification via Thurston theory ($\rightarrow$ postcritically fixed case.)

15 Interesting challenge: attracting cycles





Newton maps may have attracting cycles of any period — even for as simple polynomials as $p(z) = z^3 - 2z + 2$! The corresponding critical orbits are not connected to the chains of bubbles (the Newton graph) — so the previous classification does not apply here.

Theorem 1 (Drach, Lodge, S., Sowinski, 2018)

Every non-repelling cycle of period ≥ 2 is contained in a renormalization domain.

This is the beginning of the story: *all difficulties of Newton dynamics are actually polynomial difficulties.*

16 Newton puzzles and the Fatou–Shishikura-injection

Kostiantyn Drach, Russell Lodge, S., Maik Sowinski

Puzzles and the Fatou–Shishikura-injection for rational Newton maps; arXiv:1805.10746

Puzzles of Newton maps: iterated preimages of Newton graphs.

Main difficulty: proper containment of complementary components; needed for polynomial-like maps / renormalization.

Theorem 2 (Fatou–Shishikura-injection)

There is a dynamically natural injection from the set of non-repelling cycles to the set of critical points: every non-repelling cycle has its own critical point(s).

Idea: If a non-repelling cycle has period 1, it is the root of the polynomial and hence an attracting fixed point. All others are contained in renormalization domains, so one can use the theory of polynomials.

True in which greater generality?

Provides self-contained foundation for all subsequent Newton work.

17 Two main directions of subsequent research

A. Thurston theory: classification of all postcritically finite maps

for polynomials: every pcf polynomial has a unique Hubbard tree, and different polynomials have different trees.

B. Yoccoz theory:

a) all points in the Julia set different (trivial dynamical fibers)

b) all points in parameter space are different (trivial parameter fibers: parameter rigidity)

18 Postcritically finite maps & Thurston's theorem

Hyperbolic components in parameter space are classified through dynamics at the center: postcritically finite.

General strategy for classification of rational maps

1. (***) From holomorphic dynamical system, extract invariant tree or graph that "describes the dynamics"

2. (**) Find a classification of the resulting graphs

3. To show that these graphs are actually realized as holomorphic maps:
a) (*) the map on graph extends to postcritically finite topological branched cover on sphere (up to homotopy)
b) (***) this branched cover of sphere is realized by rational map

Difficulty in 1: hard to find good combinatorial structure for non-polynomial rational maps (until recently, no good large family): need analog to Hubbard tree

Difficulty in 3b: under which conditions can topological dynamics on sphere be promoted to rational map? (Thurston)

First main difficulty: for postcritically finite Newton map, need invariant graph that describes dynamics of all critical points.

a) The *Newton graph* (component of appropriate backwards image of channel diagram). Captures dynamics of all fixed points and their preimages, no higher order periodic points. Describes the global structure of Newton dynamics (structure of bubbles).

b) *Embedded Hubbard trees* capture dynamics of all (pre)periodic critical points of eventual period ≥ 2

c) Newton rays connect the embedded Hubbard trees to the Newton graph: chain of infinitely many bubbles that start at immediate basins and converge to ("land at") the embedded Hubbard trees. — Natural construction but involving choice (like Poirier's "left or right" supporting rays for polynomials.

Russell Lodge, Yauhen Mikulich, S., *Combinatorial properties of Newton maps*. arXiv:1510.02761.

Extended Newton graphs: the union of Newton graph, embedded Hubbard trees, and the Newton rays connecting them.

Provides branched cover in sense of Thurston:

a) the dynamics on the graph extends uniquely to a branched cover of the sphere;

b) it describes the topological dynamics of the Newton map uniquely up the homotopy rel postcritical set.

Second main difficulty: describe the resulting trees (next slide) and classify which of them are coming from rational maps (that are automatically Newton maps)

Russell Lodge, Yauhen Mikulich, S.: *A classification of postcritically finite Newton maps*. arXiv:1510.02771.

First large non-polynomial family of rational maps that is classified via Thurston theory.

21 Extended Newton graphs

DEFINITION 7.3 (Abstract extended Newton graph). Let $\Sigma \subset \mathbb{S}^2$ be a finite connected graph, and let $f : \Sigma \to \Sigma$ be a weak graph map. A pair (Σ, f) is called an *abstract extended Newton graph* if the following are satisfied:

(1) (Edges and vertices) Every edge must be one of the following three types (defined respectively in items (2),(3-4), and (6-7) below):
 - Type N: An edge in the abstract Newton graph Γ
 - Type H: An edge in a periodic or pre-periodic abstract Hubbard tree
 - Type R: A periodic or pre-periodic abstract Newton ray with respect to (Γ, f).

 As a consequence, every vertex of Σ is either a Hubbard tree vertex or a Newton graph vertex.

(2) (Abstract Newton graph) There exists a positive integer N and an abstract Newton graph Γ at level N so that $\Gamma \subseteq \Sigma$. Furthermore N is minimal so that condition (5) holds.

(3) (Periodic Hubbard trees) There is a finite collection of (possibly degenerate) minimal abstract extended Hubbard trees $H_i \subset \Sigma$ which are disjoint from Γ, and for each H_i there is a minimal positive integer $m_i \geq 2$ called the *period of the tree* such that $f^{m_i}(H_i) = H_i$.

(4) (Preperiodic trees) There is a finite collection of possibly degenerate trees $H'_i \subset \Sigma$ of preperiod ℓ_i, i.e. there is a minimal positive integer ℓ_i so that $f^{\ell_i}(H'_i)$ is a periodic Hubbard tree (H'_i is not necessarily a Hubbard tree). Furthermore for each i, the tree H'_i contains a critical or postcritical point.

(5) (Trees separated) Any two different periodic or pre-periodic Hubbard trees lie in different complementary components of Γ.

(6) (Periodic Newton rays) For every periodic abstract extended Hubbard tree H_i of period m_i, there is exactly one periodic abstract Newton ray $\mathcal{R}_i$ connecting H_i to Γ. The ray lands at a repelling fixed point $\omega_i \in H_i$ and has period $m_i \cdot r_i$ where r_i is the period of any external ray landing at the corresponding fixed point of the polynomial realizing H_i.

(7) (Preperiodic Newton rays) For every preperiodic tree H'_i, there exists at least one preperiodic abstract Newton ray in Σ connecting H'_i to Γ. A preperiodic ray landing at a periodic Hubbard tree must have preperiod 1.

(8) (Unique extendability) For every vertex $y \in V(\Sigma)$ and every component U of $\mathbb{S}^2 \setminus \Sigma$, the local extension $\tilde{f}$ from (3.1) is injective on $\bigcup_{v \in f^{-1}(y)} U_v \cap U$.

(9) (Topological admissibility) $\sum_{x \in V(\Sigma)} (\deg_x f - 1) = 2d_\Gamma - 2$, where d_Γ is the degree of the abstract channel diagram $\Delta \subset \Gamma$.

Axiomatic description of Extended Newton graphs leads to family of branched covers. Can they be obstructed?

Key idea: *Arcs Intersecting Obstructions Theorem* by Tan Lei / Kevin Pilgrim.

Channel diagram (Part of Newton graph that connects roots to ∞) provides invariant arc system in their sense.

Two possibilities:
a) invariant multicurve intersects channel diagram;
b) it does not.

In case a), useTan/Pilgrim theorem to show that curve must surround ∞ only: not essential, not an obstruction

In case b) that theorem implies that obstruction is disjoint from entire Newton graph: can only intersect little Hubbard trees. So the global Newton dynamics is "only as obstructed as embedded polynomial dynamics". If only non-obstructed Hubbard trees used, then global dynamics non-obstructed.

23 Yoccoz theory for Newton maps

Kostiantyn Drach, S., *Rigidity of Newton dynamics*. arxiv 1812.11919.

Theorem 3 (The Rational Rigidity Principle for Newton maps)

The fine structure of Newton maps, both in dynamics and parameter space, is "trivial" except where embedded polynomial dynamics interferes.

This means:

a) "points are points" in the Julia set of *every polynomial Newton map*, everywhere except when renormalizable

b) any two polynomial Newton maps are combinatorially rigid modulo renormalization: it both have the same combinatorial structure (the bubbles are arranged in the same way); and if they are renormalizable, then the renormalizations are hybrid equivalent and at the same combinatorial location, then the maps are conformally conjugate.

(Related work in non-renormalizable setting by Roesch, Yin, Zeng.)

24 Some outlook

1. What happens beyond Newton maps?

Decomposition theorem of rational maps (with Dima Dudko and Mikhail Hlushchanka; in writing) *Every postcritically finite rational maps decomposes naturally into Newton-like components (to which our theory applies) and Sierpinski-like components*.

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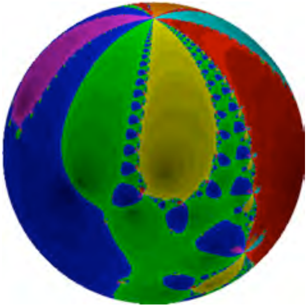
2. Dedicated talk on *Rational Rigidity for Newton Maps* by Kostiantyn Drach right after this one.



Thank you for your attention.