Distortion and Distribution of Sets under Inner Functions

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Inner Functions

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- Invariant Subspaces
  $\mathbb{H}^2 = \{g(z) = \sum_{n\geq 0} a_n z^n : \sum |a_n|^2 < \infty\}$.

$$S : \mathbb{H}^2 \to \mathbb{H}^2$$
$$g(z) \mapsto z\, g(z)$$

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Theorem (Beurling, 49)

M subspace of $\mathbb{H}^2$.

$$SM \subseteq M \iff M = f\mathbb{H}^2 \text{ for some } f \text{ inner.}$$

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Dynamics

- $\Omega \subsetneq \mathbb{C}$ simply connected
  $g : \Omega \to \Omega$ analytic
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  Dynamics of $g \longleftrightarrow$ Dynamics of f.
- If $g : \mathbb{C} \to \mathbb{C}_\infty$ meromorphic and Ω is an invariant Fatou component, then f is inner. (Baranski, Fagella, Jarque, Karpinska)

Examples

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- Finite Blaschke products. Given $z_1, \ldots, z_N \in \mathbb{D}$

$$f(z) = \prod_{k=1}^{N} \frac{z - z_k}{1 - \overline{z_k} z}, \quad z \in \mathbb{D}.$$

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- Infinite Blaschke products. Given $\{z_k\} \subset \mathbb{D}$, $\sum(1 - |z_k|) < +\infty$,

$$B(z) = \prod_{k=1}^{\infty} \frac{-\overline{z_k}}{|z_k|} \frac{z - z_k}{1 - \overline{z_k} z}, \quad z \in \mathbb{D}.$$

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- Singular Inner Functions. Given a positive singular measure μ on $\partial\mathbb{D}$,

$$S_\mu(z) = \exp\left(-\int_{\partial\mathbb{D}} \frac{\xi + z}{\xi - z}\, d\mu(\xi)\right), \quad z \in \mathbb{D}.$$

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Theorem

f inner. Then, $f=BS_\mu$.

Singularities

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$0 - 1$ Law

Let $\xi \in \partial\mathbb{D}$. Either

(a) *There exists an arc J, $\xi \in J$, such that f extends analytically across J or*

(b) *For every arc J, $\xi \in J$, $\overline{f(J \setminus \{\xi\})} = \partial\mathbb{D}$.*

Distortion

Definition

For $z \in \mathbb{D}$,

$$w_z(E) = \frac{1}{2\pi}\int_E \frac{1-|z|^2}{|\xi - z|^2}\,|d\xi|, \qquad E \subset \partial\mathbb{D}.$$

$$\begin{aligned} w_z &= \textit{harmonic measure from } z \\ w_0 &= \textit{Lebesgue measure on } \partial\mathbb{D} \end{aligned}$$

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Theorem (Lowner)

f *inner,* $z \in \mathbb{D}$. *Then,*

$$w_z(f^{-1}(E)) = w_{f(z)}(E), \qquad E \subset \partial\mathbb{D}.$$

If $z = f(z) = 0$, $|f^{-1}(E)| = |E|$, $E \subset \partial\mathbb{D}$.

Distortion

Definition

For $0 < \alpha < 1$ *and* $z \in \mathbb{D}$,

$$M_\alpha(w_z)(E) = \inf\{\sum w_z(J_k)^\alpha : E \subset \cup J_k\} \quad E \subset \partial\mathbb{D}$$

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Theorem (Fernandez, Pestana, 92)

f *inner*, $0 < \alpha < 1$ *and* $z \in \mathbb{D}$. *Then*

$$M_\alpha(w_z)(f^{-1}(E)) \geq C_\alpha M_\alpha(w_{f(z)}(E)), \qquad E \subset \partial\mathbb{D},$$

(and, consequently, $\dim f^{-1}(E) \geq \dim E$, *for any* $E \subset \partial\mathbb{D}$*)*
If $z = f(z) = 0$, $M_\alpha(f^{-1}(E)) \geq C_\alpha M_\alpha(E)$,$E \subset \partial\mathbb{D}$.

Distortion with respect to a boundary point

Definition

$f : \mathbb{D} \to \mathbb{D}$ *analytic and* $p \in \partial\mathbb{D}$. *We say* $|f'(p)| < \infty$ *if* $f(p) = \lim_{r\to 1} f(rp) \in \partial\mathbb{D}$ *exists (p is a Boundary Fatou point) and*

$$f'(p) = \lim_{\Gamma \ni z \to p} \frac{f(z) - f(p)}{z - p} \text{ exists.}$$

Otherwise $|f'(p)| = \infty$.

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- $J \subset \partial\mathbb{D}$ arc. $\mu_p(J) < \infty \iff p \notin \overline{J}$.
- μ_p measures the size of E and the distribution of E around p.
- $E = \cup J_k$. Then, $\mu_p(E) < \infty \iff \sum \frac{|J_k|}{\text{dist}(p, J_k)^2} < \infty$.

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Theorem (Levi, N., Soler, 18)

f inner, $p \in \partial\mathbb{D}$ a BFP. Then,

$$\mu_p(f^{-1}(E)) = |f'(p)|\mu_{f(p)}(E), \qquad E \subset \partial\mathbb{D}.$$

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Extreme cases.

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Definition

$0 < \alpha < 1$ *and* $p \in \partial\mathbb{D}$, $\quad M_\alpha(\mu_p)(E) = \inf\{\sum \mu_p(I_j)^\alpha : E \setminus \{p\} \subset \cup I_j\}$

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f inner, $p \in \partial\mathbb{D}$ a BFP, $0 < \alpha < 1$. Then,

$$M_\alpha(\mu_p)(f^{-1}(E)) \geq C_\alpha |f'(p)|^\alpha M_\alpha(\mu_{f(p)})(E), \qquad E \subset \partial\mathbb{D}.$$

Denjoy-Wolff Theorem

Theorem (Denjoy-Wolff)

$f : \mathbb{D} \to \mathbb{D}$ analytic, not automorphism. Then, there exists $p \in \overline{\mathbb{D}}$ such that

$$f^n = f \circ \cdots \circ f \xrightarrow[n\to\infty]{} p \text{ unif. on compacts of } \mathbb{D}.$$

Moreover, $p \in \overline{\mathbb{D}}$ is the unique fixed point of f with $|f'(p)| \leq 1$.

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$$p \equiv DWFP$$

Dynamics of $f : \partial\mathbb{D} \to \partial\mathbb{D}$?

DWFP in $\mathbb{D}$

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Theorem (Poincaré Recurrence Theorem)

f inner, $f(0) = 0$. Then $\liminf_{n\to\infty} |f^n(\xi) - \xi| = 0$ a.e. $\xi \in \partial\mathbb{D}$.

Theorem (Ergodic Theorem)

Let f inner with $f(0) = 0$. Then $(f, |\,|)$ is ergodic and $\lim_{n\to\infty} \frac{\#\{1\le k\le n\,:\, f^k(\xi)\in J\}}{n} = |J|$ for any $J \subset \partial\mathbb{D}$.

Shrinking Targets

Notation: $J(\xi_0, r) = \{\xi \in \partial\mathbb{D} : |\xi - \xi_0| < r\}$.

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Theorem (Fernández, Melián, Pestana, 07)

f inner, not automorphism, with $f(0) = 0$. Fix $\xi_0 \in \partial\mathbb{D}$ and $r_k \geq 0$ decreasing.

(a) *If $\sum r_k = \infty$, then $\lim_{n\to\infty} \frac{\#\{1\leq k\leq n \,:\, f^k(\xi)\in J(\xi_0,r_k)\}}{\sum_{k=1}^{n} r_k} = 1$ a.e. $\xi \in \partial\mathbb{D}$.*

(b) *If $\sum r_k < \infty$, then $\liminf_{n\to\infty} \frac{|f^n(\xi)-\xi_0|}{r_n} \geq 1$ a.e. $\xi \in \partial\mathbb{D}$.*

Pommerenke, 81

f inner, $f(0) = 0$, not a rotation. Then,

$$\left| \frac{|B \cap f^{-n}(A)|}{|A|} - |B| \right| \leq C_1 e^{-C_2 n},$$

for any $A, B \subset \partial\mathbb{D}$.

DWFP in $\partial\mathbb{D}$

Theorem (Doering-Mañe, 91)

f inner with DWFP $p \in \partial\mathbb{D}$. Then
$\mu_p(f^{-1}(E)) = |f'(p)|\mu_p(E)$, *for any* $E \subset \partial\mathbb{D}$.

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Theorem (Aaranson, 81)

f inner with DWFP $p \in \partial\mathbb{D}$. Then,

(a) $f : \partial\mathbb{D} \to \partial\mathbb{D}$ *recurrent* $\iff \sum(1 - |f^n(0)|) = \infty$

(b) $f^n(\xi) \to p$ *a.e.* $\xi \in \partial\mathbb{D} \iff \sum(1 - |f^n(0)|) < \infty$

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f inner with DWFP $p \in \partial\mathbb{D}$. Then, for any neighborhood J of p,

$$\lim_{n\to\infty} \frac{\#\{1 \le k \le n \ : \ f^k(\xi) \in J\}}{n} = 1 \qquad a.e.\ \xi \in \partial\mathbb{D}$$

f inner with DWFP $p \in \partial\mathbb{D}$. Then $\alpha_k = f^k(0)/|f^k(0)| \to p$. Denote $J_k(M) = \{\xi \in \partial\mathbb{D} \, : \, |\xi - \alpha_k| \leq M\}$.

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Theorem (Levi, N., Soler)

f inner with DWFP $p \in \partial\mathbb{D}$. Then

(a) *Let* $\frac{M_n}{1-|f^n(0)|} \to \infty$. *Then,*

$$\lim_{n\to\infty} \frac{1}{n} \int_{\partial\mathbb{D}} \#\{1 \le k \le n \ : \ f^k(\xi) \in J_k(M_k)\}|d\xi| = 1.$$

(b) *Let* $\frac{\varepsilon_n}{1-|f^n(0)|} \to 0$. *Then,*

$$\lim_{n\to\infty} \frac{1}{n} \int_{\partial\mathbb{D}} \#\{1 \le k \le n \ : \ f^k(\xi) \in J_k(\varepsilon_k)\}|d\xi| = 0.$$

Recall $J_k(M) = \{\xi \in \partial\mathbb{D} : \left|\xi - \frac{f^k(0)}{|f^k(0)|}\right| \leq M\}$.

Theorem (Levi, N., Soler)

f inner with DWFP $p \in \partial\mathbb{D}$. Assume

$$\sum (1 - |f^n(0)|) < \infty$$

(a) *Let $M_n \geq 0$ with $\sum \frac{1-|f^n(0)|}{M_n} < \infty$. Then, for a.e. $\xi \in \partial\mathbb{D}$, there exists $n_0 > 0 \ : \ f^n(\xi) \in J_n(M_n)$ for any $n \geq n_0$.*

(b) *Let $\varepsilon_n \geq 0$ with $\sum \frac{\varepsilon_n}{1-|f^n(0)|} < \infty$. Then*

$$|\{\xi \in \partial\mathbb{D} \ : \ f^n(\xi) \in J_n(\varepsilon_n) \text{ for infinitely many } n\}| = 0.$$