Dynamics of Schwarz reflections: mating rational maps with groups

(Joint with Seung-Yeop Lee, Mikhail Lyubich, and Nikolai Makarov)

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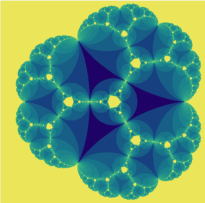
Stony Brook University

TCD 2019, Barcelona



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Definition

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1. $\sigma = \mathrm{id}$ on $\partial\Omega$.
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- Examples: Round disks, $\cdots$

Simply Connected Quadrature Domains

Proposition (Characterization of S.C.Q.D.)

A simply connected domain $\Omega \subsetneq \hat{\mathbb{C}}$ *(with* $\infty \notin \partial\Omega$ *and* $\mathrm{int}(\overline{\Omega}) = \Omega$*) is a quadrature domain if and only if the Riemann map* $\phi : \mathbb{D} \to \Omega$ *is rational.*

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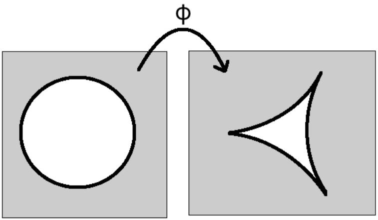
The Complement of a Deltoid as a Quadrature Domain

- The complement of the deltoid has a Riemann map $\phi(z) = z + \frac{1}{2z^2}$, so it is a quadrature domain.



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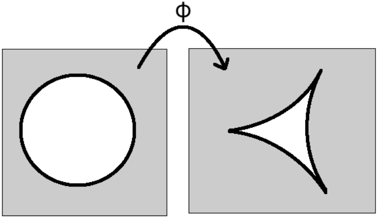
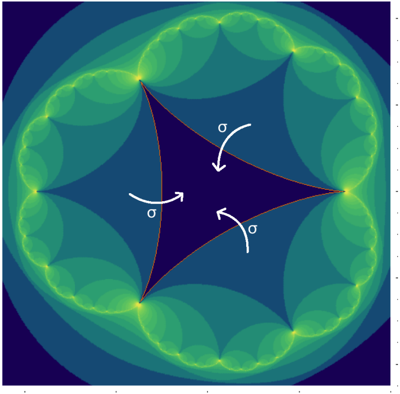


- The corresponding Schwarz reflection map σ has a unique critical point at ∞. Moreover, $\sigma(\infty) = \infty$.

Deltoid Reflection as a Mating



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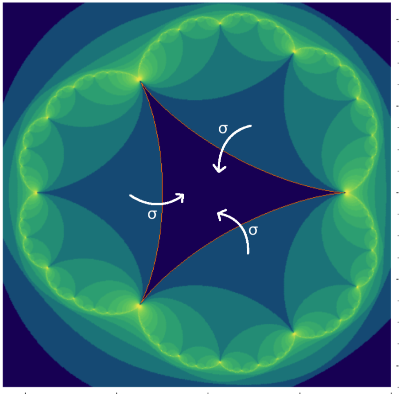

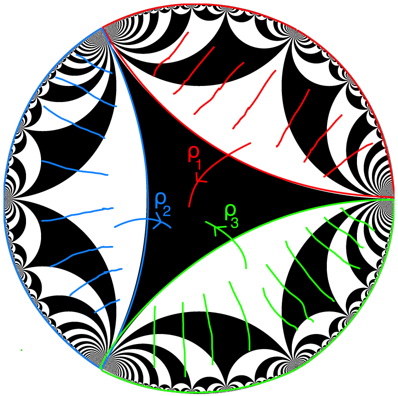

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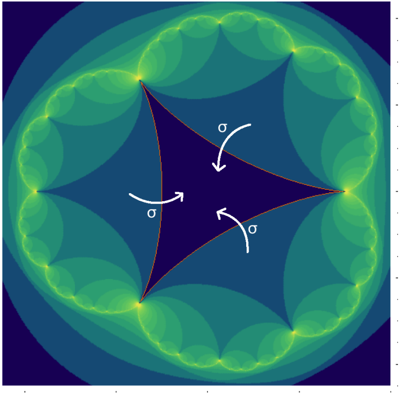

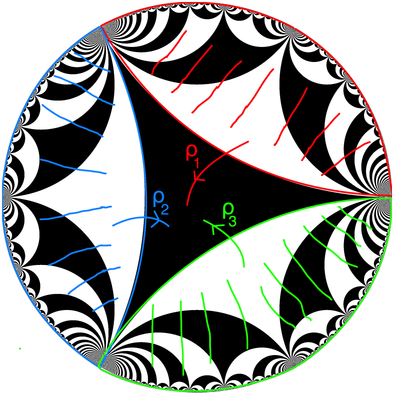

- The dynamics of the deltoid reflection map is a "mating" of ρ (on the tiling set) and $\overline{z}^2$ (on the non-escaping set).

The Welding Map

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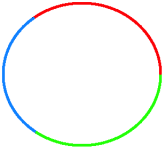

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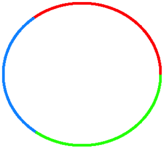

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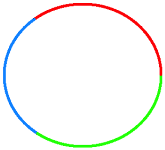

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- Consequently, ρ and $\overline{z}^2$ are topologically conjugate by a circle homeomorphism $\mathcal{H}$.
- $\mathcal{H}$ conjugates the external class of quadratic antiholomorphic polynomials and that of the ideal triangle group.

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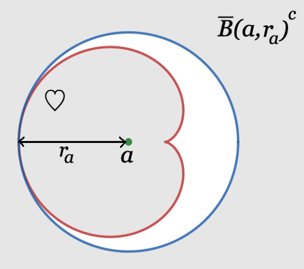

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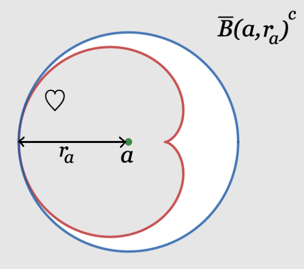

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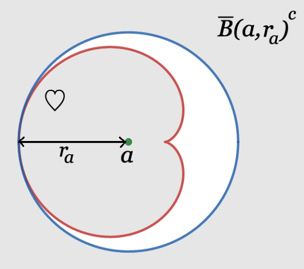

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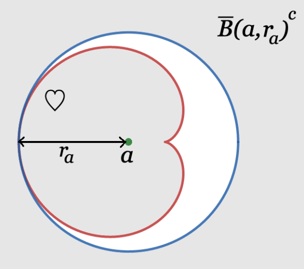

- $\Omega_a := \heartsuit \cup \overline{B}(a, r_a)^c$. We call its Schwarz reflection map F_a.
- The unique critical point of F_a is at 0.
- As a varies over the plane, we get a family of maps

$$C\&C := \{F_a : \overline{\Omega}_a \to \hat{\mathbb{C}}\}.$$

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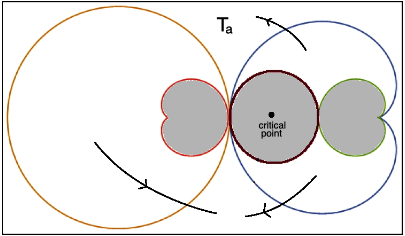
In different coordinates, F_a is a *pinched quadratic-like map*:



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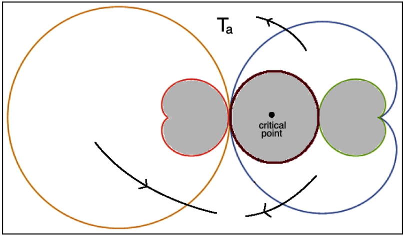

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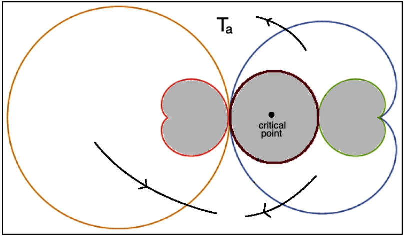

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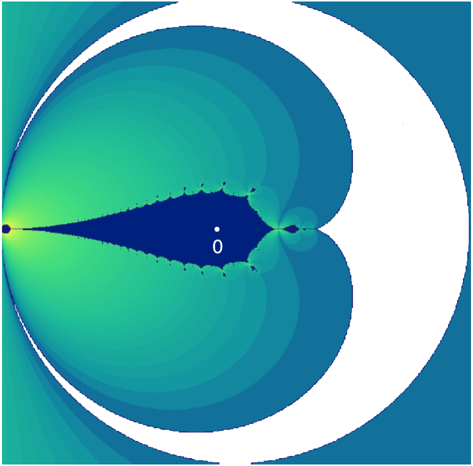
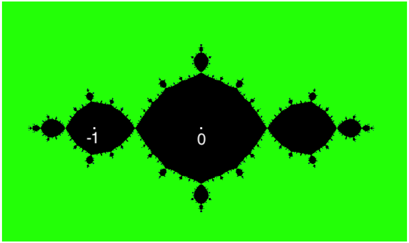
- The *non-escaping set* K_a of F_a is the complement of the tiling set. It is the *filled Julia set* of the pinched quadratic-like map.

Dynamical Plane of the Basilica Map: $a = 0$





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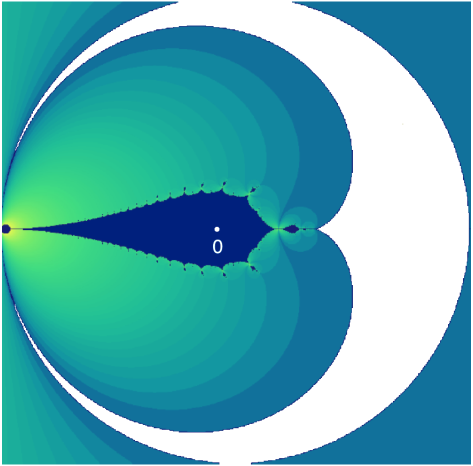

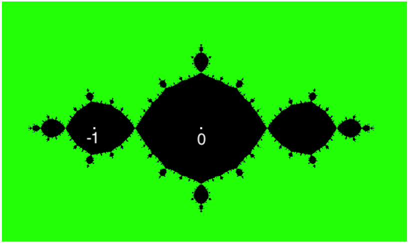

- $0 \mapsto \infty \mapsto 0$; the "Basilica" map.

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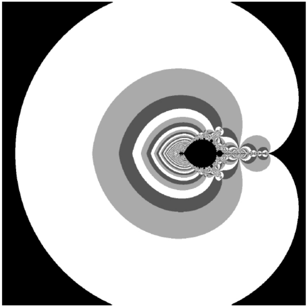

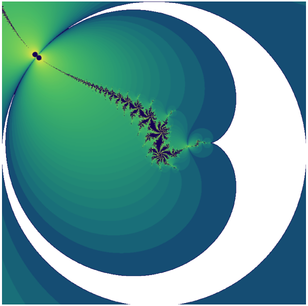

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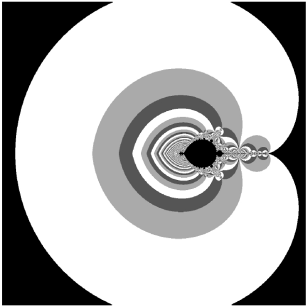

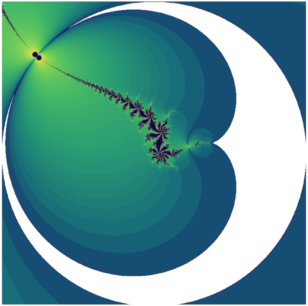

- For maps in $\mathcal{C}$, the dynamics on the tiling set is conformally conjugate to the reflection map ρ (i.e. group structure).

Bijection between Geom. Finite Parameters

Theorem (Lee, Lyubich, Makarov, M)

There exists a natural combinatorial bijection χ between the geometrically finite parameters of C&C and those in the basilica limb of the tricorn such that the laminations of the corresponding maps are related by the circle homeomorphism $\mathcal{H}$.

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- Surjectivity: Realiziing geometrically finite Schwarz maps (in $C\&C$) with prescribed laminations via "parameter rays".

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Theorem (Lee, Lyubich, Makarov, M)

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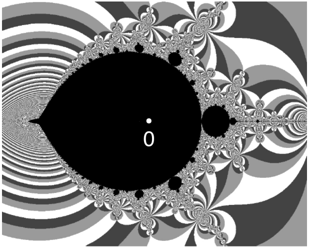

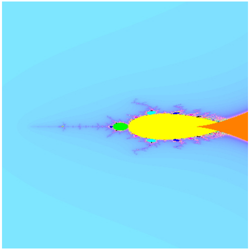

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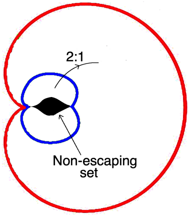

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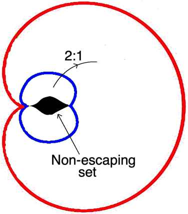

- Pinched quadratic-like maps with a unique point of pinching $\implies$ Quasiconformal straightening to *parabolic* rational maps.

Correspondences = Rational Map + Group

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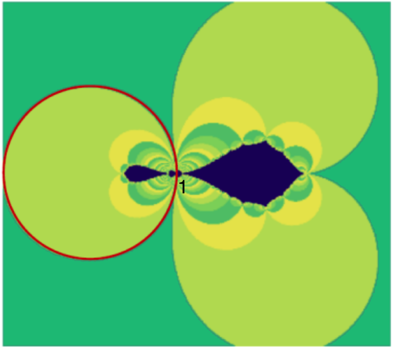

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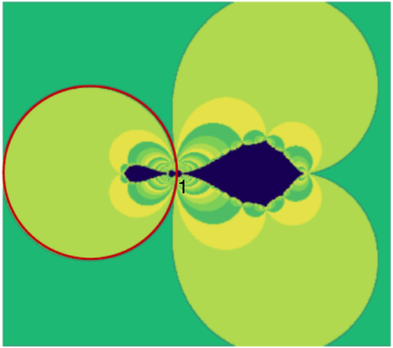

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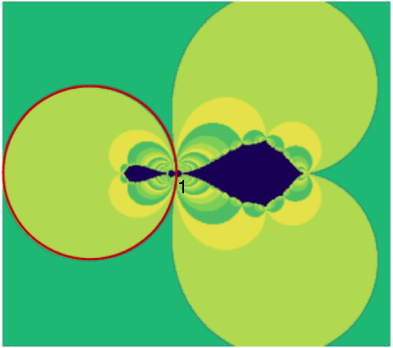

- Dynamics on the tiling set $\cong \mathbb{Z}_2 * \mathbb{Z}_3 \cong SL_2(\mathbb{Z})$.
- Dynamics on the non-escaping set $\cong$ Anti-holomorphic rational map.

Thank you!