Pressure and conformal measures for transcendental meromorphic maps

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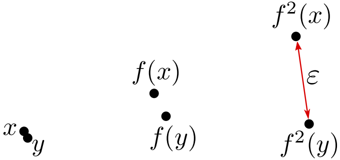
(n, ε) – separated sets

Let $f : X \to X$ be a continuous transformation of a compact metric space (X, d). Define

$$d_n(x, y) = \max\{d(f^k(x), f^k(y)) : \ 0 \le k \le n-1\}.$$

If $d_n(x, y) > \varepsilon$, then x and y are called (n, ε) – separated.



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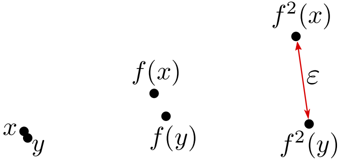

Topological entropy: $$h_{top}(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N_n(\varepsilon),$$
where $N_n(\varepsilon)$ be a maximal cardinality of (n, ε) separated set.

Pressure – general definition

Topological pressure of f with respect to the potential φ ($\varphi : X \to \mathbb{R}$ continuous function)

$$P(f,\varphi) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \sup_E \sum_{x\in E} \exp \underbrace{\left(\sum_{k=0}^{n-1} \varphi(f^k(x)) \right)}_{S_n\varphi(x)},$$

where the supremum is taken over all (n,ε)-separated sets E in X.

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Conformal repellers

$X \subset \mathbb{C}$

f – holomorphic map defined in a neighbourhood of X

$X \subset \mathcal{J}(f)$ is called a **conformal repeller** if

- X is compact
- X is f–invariant (i.e. $f(X) \subset X$)
- $\exists c > 0, \lambda > 1 \quad |(f^n)'(z)| \geq c\lambda^n$ for every $z \in X$ and every $n \geq 1$.

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Theorem (Bowen, Ruelle)

If $X \subset \mathcal{J}(f)$ is transitive, isolated, conformal repeller and φ is a Hölder continuous function in a neighbourhood of X, then

$$P(f_{|X}, \varphi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{w \in f^{-n}(z)\cap X} \exp(S_n\varphi(w)),$$

for $z \in X$.

In particular the pressure doesn't depend on z

Conformal repellers

geometric potential: $\varphi_t(z) = -t \log |f'(z)|$, for $t \geq 0$.

$$P(f_{|X}, t) := P(f_{|X}, \varphi_t)$$

Then

$$P(f_{|X}, t) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{w\in f^{-n}(z)\cap X} \exp(S_n\varphi(w))$$

$$= \lim_{n\to\infty} \frac{1}{n} \log \sum_{w\in f^{-n}(z)\cap X} |(f^n)'(w)|^{-t}$$

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Note that

$$\sum_{w \in f^{-n}(z)\cap X} |(f^n)'(w)|^{-t} \sim \sum_{w \in f_w^{-n}\cap X} (\operatorname{diam} f_w^{-n} D(z,\delta))^t,$$

where $D(z, \delta)$ is a disc of small radius.

Perron-Frobenius-Ruelle operator

$$P(f|_X, \varphi_t) = \lim_{n\to\infty} \frac{1}{n} \log \mathcal{L}^n_{\varphi_t}(1),$$

where $\mathcal{L}_{\varphi_t}$ is the operator acting on $C(X)$:

$$\mathcal{L}_{\varphi_t}(g)(z) = \sum_{w\in f^{-1}(z)\cap X} \exp(\varphi(w))g(w).$$

The operator $\mathcal{L}_{\varphi_t}$ has a unique positive eigenfunction u_φ and the dual operator has an eigenmeasure ν_φ on X (conformal measure) such that the measure μ_φ (equivalent to ν_φ) with Radon-Nikodym derivative $\frac{d\mu_\varphi}{d\nu_\varphi} = u_\varphi$ is the f-invariant measure on X [Ruelle, Bowen,Walters $\sim$ 1970].

Bowen's formula

Theorem (Bowen, Ruelle $\sim$ 1979)

If X is transitive, isolated, conformal repeller then

$$\dim_H(X) = t_0,$$

where t_0 is the unique zero of the pressure function

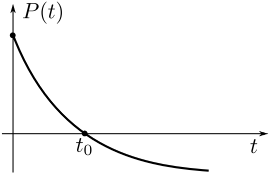
$$t \mapsto P(f|_X, t) = \lim_{n\to\infty} \frac{1}{n} \log \underbrace{\sum_{w\in f^{-n}(z)\cap X} |(f^n)'(w)|^{-t}}_{S_n}, \quad z \in X, t \geq 0$$

The limit exists and doesn't depend on z, so the pressure is a function of t.

Bowen's formula

$$P(t) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{w\in f^{-n}(z)\cap X} |(f^n)'(w)|^{-t}$$

- continuous
- strictly decreasing
- $P(0) > 0$
- $\lim_{t\to+\infty} P(t) = -\infty$



For hyperbolic rational functions $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ the whole Julia set is a conformal repeller. A rational map f is hyperbolic if $\overline{\mathcal{P}(f)} \cap \mathcal{J}(f) = \emptyset$.

Bowen's formula for rational maps

f hyperbolic rational $\implies$ $\dim_H(\mathcal{J}(f)) = t_0$.

Bowen's formula for transcendental maps

For transcendental maps the pressure can be infinite for all t, even if the map is hyperbolic, e.g. for $E_\lambda(z) = \lambda \exp z$, $\lambda \in \mathbb{C} \setminus \{0\}$.

Definition

A meromorphic map $f : \mathbb{C} \to \overline{\mathbb{C}}$ is called hyperbolic if $\overline{\mathcal{P}(f)} \cap \mathcal{J}(f) = \emptyset$, where

$$\mathcal{P}(f) = \bigcup_{k=0}^{\infty} f^k(\mathrm{Sing}(f)), \qquad \mathrm{Sing}(f) - \text{ critical and asymptotic values.}$$

Urbański and Zdunik considered the pressure function for the map $\widetilde{E_\lambda} : \mathbb{C}/2\pi i\mathbb{Z} \to \mathbb{C}/2\pi i\mathbb{Z}$, which is obtained from E_λ by identifying points that differ by $2\pi i$.

Bowen's formula for exponentials (Urbański and Zdunik, 2003)

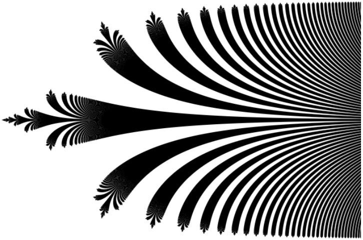
If E_λ is hyperbolic, then $\dim_H(\mathcal{J}_r(\mathrm{E}_\lambda)) = t_0$, where t_0 is the unique zero of the pressure function for $\widetilde{E_\lambda}$.

Exponential family $E_\lambda(z) = \lambda \exp(z)$

$\mathcal{J}_r(E_\lambda)$ - radial Julia set
= the set of non-escaping points

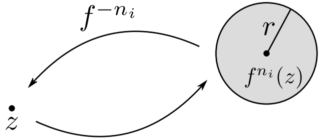
Urbański and Zdunik proved that

$dim_H(\mathcal{J}_r(E_\lambda)) < 2$
(less than $dim_H(\mathcal{J}(E_\lambda))$).



Radial Julia set

$\mathcal{J}_r(f)$ is the set of all $z \in \mathcal{J}(f)$ with the property:
there exists $r = r(z) > 0$ and $n_j \to \infty$, such that a holomorphic branch of f^{-n_j} sending $f^{n_j}(z)$ to z is well defined on the spherical disc $D(f^{n_j}(z), r)$.



- If f is entire and hyperbolic, then
  $\mathcal{J}_r(f) = \{z \in \mathcal{J}(f) : f^n(z) \not\to \infty \text{ as } n \to \infty\} = \mathcal{J}(f) \setminus \mathcal{I}(f)$,
  where $\mathcal{I}(f)$ - escaping set.
- If f has a finite number of poles, then
  $\mathcal{J}_r(f) \subset \mathcal{J}(f) \setminus (\mathcal{I}(f) \cup \bigcup_{n=1}^{\infty} f^{-n}(\infty))$.
- If f is hyperbolic, then $\mathcal{J}(f) \setminus (\mathcal{I}(f) \cup \bigcup_{n=1}^{\infty} f^{-n}(\infty)) \subset \mathcal{J}_r(f)$.

Pressure for meromorphic maps in class $\mathcal{B}$

$$\mathcal{S} = \{f \text{ meromorphic} : \ \mathit{Sing}(f) \text{ is finite}\}$$

$$\mathcal{B} = \{f \text{ meromorphic} : \ \mathit{Sing}(f) \text{ is bounded}\}$$

Gwyneth Stallard in
The Hausdorff dimension of Julia sets of hyperbolic meromorphic functions
Math.Proc.Cambridge Philos.Soc. 1999
considered

$$S_n(t,z) = \sum_{w \in f^{-n}(z)} \frac{1}{|(f^n)^*(w)|^t},$$

where $*$ denotes the derivative with respect to the spherical metric

$$|ds| = \frac{2\,|dz|}{1+|z|^2}, \qquad \text{that is } |f^*(z)| = \frac{(1+|z|^2)|f'(z)|}{1+|f(z)|^2}.$$

Pressure for meromorphic maps in class $\mathcal{B}$

Let

$$\boxed{P(f,t,z) = \lim_{n\to\infty} \frac{1}{n} \ln S_n(t,z)}, \quad \text{where} \quad S_n(t,z) = \sum_{w \in f^{-n}(z)} \frac{1}{|(f^n)^*(w)|^t}.$$

If the limit exists, then we call $P(f,t,z)$ the topological pressure for f at the point z (called also the tree pressure).

The hyperbolic pressure $P_{hyp}(t)$ is the supremum of the pressures $P(f|_X, t)$ over all transitive isolated conformal repellers $X \subset \mathcal{J}(f)$

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- Does $P(f,t,z)$ depend on z (if exists)?
- $P(f,t,z) = P_{hyp}(t)$?

Pressure for meromorphic maps in class $\mathcal{S}$ and $\mathcal{B}$

Theorem A (Barański, Zdunik, K., 2012)

- *Let $f \in \mathcal{S}$. Then there exists a set E, $\dim_H(E) = 0$, such that for every $z \in \mathbb{C} \setminus E$ and for every $t > 0$ the topological pressure $P(f, t, z)$ exists (can be $+\infty$), it doesn't depend on z and*

$$P(t) = P_{hyp}(t).$$

- *Let $f \in \mathcal{B}$ be hyperbolic map. Then for every $z \in \mathcal{J}(f)$ and for every $t > 0$ the topological pressure $P(f, t, z)$ exists (can be $+\infty$), it doesn't depend on z and $P(t) = P_{hyp}(t)$.*
- *The same holds for nonhyperbolic maps in $\mathcal{B}$ ("nonexceptional") and $z \in \mathcal{J}(f) \setminus \overline{\mathcal{P}(f)}$.*

The set E consists of z which are well approximated by trajectories of singular points.

Bowen's formula for meromorphic maps in class $\mathcal{B}$

The function $t \mapsto P(t)$ is non-increasing for $t > 0$, convex (when it is finite) and $P(2) \leq 0$.

Define $\delta(f) = \inf\{t > 0 : P(t) \leq 0\}.$

Corollary

$$\dim_{hyp} \mathcal{J}(f) = \delta(f),$$

where $\dim_{hyp} \mathcal{J}(f) = \sup\{\dim_H(X) : \ X \subset \mathcal{J}(f), \ X - \textit{conf. repeller}\}.$

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L. Rempe, 2008

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Bowen's formula for meromorphic maps in class $\mathcal{B}$

If $f \in \mathcal{B}$ non-exceptional and s.t. $\mathcal{J}(f) \setminus \overline{\mathcal{P}(f)} \neq \emptyset$, then $\dim_H \mathcal{J}_r(f) = \delta(f)$.

Conformal measures

Definition

A Borel probability measure ν on $\mathcal{J}(f)$ is called **t − conformal** if

$$\nu(f(A)) = \int_A |f'(z)|^t d\nu(z)$$

for every Borel set $A \subset \mathbb{C}$ on which f is injective

- If f is rational ($\deg f \geq 2$), then the Julia set of f always admits a conformal measure, and the minimal exponent t for which such a measure exists is equal to $\dim_H(\mathcal{J}_r(f))$ (Przytycki, 1999).
- For transcendental maps the question of existence of a t-conformal measure, where $t = \dim_H(\mathcal{J}_r(f))$, was answered positively for several specific families of maps (Urbański and Zdunik, Mayer and Urbański)

Conformal measures and zero of the pressure

We consider a t-conformal measure on $\mathcal{J}(f)$ with respect to the spherical metric, denote it by m_t. Then m_t satisfies $m_t(f(A)) = \int_A |f^*(z)|^t dm_t(z)$ for every Borel set $A \subset \mathbb{C}$ on which f is injective.

Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a transcendental meromorphic function for which $P(f,t)$ exists.

Question

Is the existence of a value $t > 0$ such that $P(f,t) = 0$ equivalent to the existence of a t-conformal measure on $\mathcal{J}(f)$?

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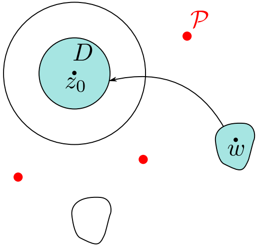
Theorem B (Barański, Zdunik, K., 2017)

If a hyperbolic transcendental meromorphic map $f : \mathbb{C} \to \overline{\mathbb{C}}$ admits a t-conformal measure m_t on $\mathcal{J}(f)$ for some $t > 0$, with respect to the spherical metric, then $P(f,t) \leq 0$.
Moreover, if $m_t(\mathcal{J}(f) \setminus \mathcal{I}(f)) > 0$, then $P(f,t) = 0$.

Idea of the proof

$\exists\ m_t$ conformal s.t. $m_t(\mathcal{J}(f) \setminus \mathcal{I}(f)) > 0 \implies P(f,t) = 0$



By conformality and bounded (spherical) distortion

$$m_t(f^{-n}(D)) \asymp m_t(D) \sum_w \frac{1}{|(f^n)^*(w)|^t}$$

Hence $S_n(t,z_0) = \displaystyle\sum_{w \in f^{-n}(z_0)} \frac{1}{|(f^n)^*(w)|^t}$ is finite and

$$P(f,t) = \lim_{n\to\infty} \frac{1}{n} \ln S_n(t,z_0) \leq 0.$$

Idea of the proof

$\exists\ m_t$ conformal s.t. $\ m_t(\mathcal{J}(f) \setminus \mathcal{I}(f)) > 0 \implies P(f,t) = 0$

Assume that $P(f,t) < 0$. Then for every D and large n

$$\sum_{w \in f^{-n}(z_0)} \frac{1}{|(f^n)^*(w)|^t} < \mathrm{e}^{-n\delta}.$$

Then $m_t(f^{-n}(D)) \asymp m_t(D) \sum_w \frac{1}{|(f^n)^*(w)|^t}$ implies that

$$m_t(f^{-n}(D)) < c\mathrm{e}^{-n\delta}.$$

Then $f^n(z) \in \mathbb{C} \setminus D$ for m_t – almost every z and sufficiently large n.

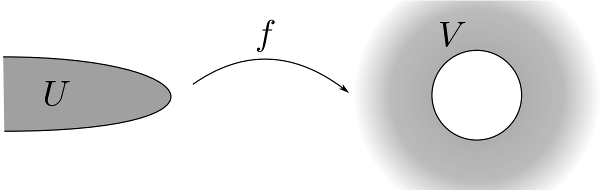
Hence for m_t – almost every $z \quad f^n(z) \to \overline{\mathcal{P}(f)} \cup \{\infty\}$ as $n \to \infty$.

Conformal measures and zero of the pressure

Theorem C (Barański, Zdunik, K., 2017)

Let f be a transcendental meromorphic map with a logarithmic tract over ∞. Assume that $f \in \mathcal{S}$, or f is a non-exceptional map from $\mathcal{B}$ such that $\mathcal{J}(f) \setminus \overline{\mathcal{P}(f)} \neq \emptyset$.
If $P(f,t) = 0$ for some $t > 0$, then there exists a t-conformal measure m_t on $\mathcal{J}(f)$, with respect to the spherical metric. Moreover,

$$m_t(\mathbb{C} \setminus \mathbb{D}(r)) = o\left(\frac{(\ln r)^{3t}}{r^t}\right) \quad \textit{as} \quad r \to \infty.$$



Conformal measures and zero of the pressure

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Is m_t supported on $\mathcal{J}_r(f)$?

Conformal measures and zero of the pressure

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Is m_t supported on $\mathcal{J}_r(f)$?

V.Mayer, A.Zdunik, 2019

There exists a hyperbolic (disjoint type) entire function $f \in \mathcal{B}$ such that

$$\delta(f) = \theta \in (1, 2)$$

and such that f doesn't have a conformal measure on its radial Julia set.

Construction of conformal measure

Assume that $P(f,t)=0$. Then $e^{-n\varepsilon} < S_n(t) < e^{n\varepsilon}$.

Take $s>0$. Then the series $\sum_{n=1}^{\infty} e^{-ns} S_n(t)$ is convergent.

Define

$$\mu_s = \frac{1}{\sum_s} \sum_{n=1}^{\infty} b_n e^{-ns} \sum_{w \in f^{-n}(z)} \frac{\delta_w}{|(f^n)^*(w)|^t},$$

where b_n is such that $b_n/b_{n+1} \to 1$
and

$$\lim_{s\to 0^+} \underbrace{\sum_{n=1}^{\infty} b_n e^{-ns} S_n(t)}_{\sum_s} = +\infty.$$

The measure μ_s is supported on $\mathcal{J}(f)$. If the family μ_s is tight, then

$$m_t = \lim_{j\to\infty} \mu_{s_j}, \quad \text{for some} \quad s_j \to 0^+.$$

Tools used in the proof of tightness

The main idea is to compare $S_{n+1}(t,z)$ and $S_n(t,z)$.
For a set $A \subset \mathbb{C}$ denote

$$S_n^A(t,z) = \sum_{w \in f^{-n}(z) \cap A} |(f^n)^*(w)|^{-t}.$$

We prove that for $z \in \mathcal{J}(f) \setminus \overline{\mathcal{P}(f)}$

$$S_{n+1}(t,z) \geq c \sum_{k=k_0}^{\infty} \frac{2^{kt}}{k^{3t}} S_n^{\mathbb{D}(2^{k+1}) \setminus \mathbb{D}(2^k)}(t,z).$$

Using this fact we show that

$$\sum_{k=1}^{\infty} \frac{2^{kt}}{k^{3t}} \mu_s(\mathcal{J}(f) \setminus \mathbb{D}(2^k)) < \infty,$$

which implies that

$$\mu_s(\mathcal{J}(f) \setminus \mathbb{D}(2^k)) < c \frac{k^{3t}}{2^{kt}}.$$

Thank you for your attention!