Linear skew-products and fractalization

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Introduction

Introduction

Consider

$$\left.\begin{array}{rcl} \bar{x} & = & f_\mu(x,\theta), \\ \bar{\theta} & = & \theta+\omega, \end{array}\right\}$$

where $x \in \mathbb{R}^n$, $\theta \in \mathbb{T}^1$, $\mu \in \mathbb{R}$ is a parameter, $\omega \in (0, 2\pi) \setminus 2\pi\mathbb{Q}$ and f_μ is smooth enough.

Assume that, for a given μ_0, there is an attracting invariant curve, $x_{\mu_0}(\theta)$ with rotation number ω,

$$f_{\mu_0}(x_{\mu_0}(\theta),\theta) = x_{\mu_0}(\theta+\omega), \qquad \forall\, \theta \in \mathbb{T}^1.$$

We are interested in the continuation of x_{μ_0} w.r.t. the parameter μ.

Example: the quasiperiodically forced logistic map

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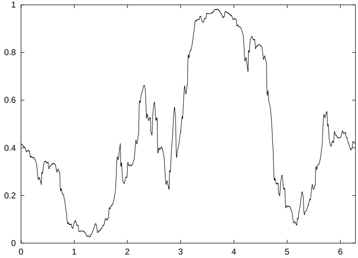
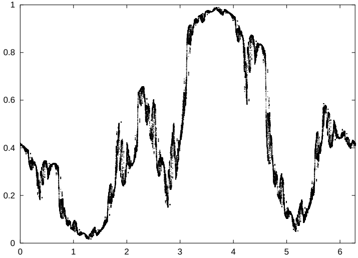
$$\left.\begin{array}{rcl} \bar{x} & = & \alpha(1+\varepsilon\cos(\theta))x(1-x), \\ \bar{\theta} & = & \theta+\omega, \end{array}\right\}$$

with $\omega = \pi(\sqrt{5}-1)$ and $\varepsilon = 0.5$.





Left: $\alpha = 2.65$, $\Lambda \approx -0.03884$. Right: $\alpha = 2.665$, $\Lambda \approx -0.00845$.

In this talk we will focus on this fractalization process.

Note that to study this process by means of direct numerical simulation is a very difficult problem (see, for instance, the previous examples).

We will introduce the problem by discussing first the 1D case ($x \in \mathbb{R}$) and then we will focus on some aspects of the complex case ($x \in \mathbb{C}$).

Linear behaviour

As $f_0(u_0(\theta)+h,\theta) = f_0(u_0(\theta),\theta) + D_x f_0(u_0(\theta),\theta)h + \cdots$, the linearized dynamics around $u_0(\theta)$ is given by

$$\left.\begin{array}{rcl} \bar{x} & = & a(\theta)x, \\ \bar{\theta} & = & \theta+\omega, \end{array}\right\} \tag{1}$$

where $a(\theta) = D_x f_0(u_0(\theta),\theta)$.
In what follows, we will assume that $a(\theta) \not\equiv 0$.

Definition

(1) is called *reducible* iff there exists a (at least continuous) linear change of variables $x = c(\theta)y$ such that (1) becomes

$$\left.\begin{array}{rcl} \bar{y} & = & by, \\ \bar{\theta} & = & \theta+\omega, \end{array}\right\}$$

where b does not depend on θ.

Proposition

Assume that ω satisfies a Diophantine condition,

$$|q\omega - 2\pi p| \geq \frac{\gamma}{|q|^\tau}, \qquad \textit{for all } (p,q) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}),$$

and that a is C^∞. Then, (1) is reducible iff a has no zeros.

This result also holds if $a \in C^r$, for r big enough but, due to the effect of the small divisors, the reducing transformation does not need to be C^r.

The Lyapunov exponent of (1) at θ is

$$\lambda(\theta) = \limsup_{n\to\infty} \frac{1}{n} \ln \left| \prod_{j=0}^{n-1} a(\theta + j\omega) \right| .$$

We define

$$\Lambda = \frac{1}{2\pi} \int_0^{2\pi} \ln |a(\theta)| \, d\theta.$$

If Λ is finite, then the Birkhoff ergodic theorem implies that

$$\lambda(\theta) = \Lambda, \qquad \text{for Lebesgue-a.e. } \theta \in \mathbb{T}^1.$$

The value Λ is usually known as the Lyapunov exponent of the skew product.

Proposition

If $a(\theta)$ is C^0 and the skew product is reducible, then the Lyapunov exponent at θ, $\lambda(\theta)$, does not depend on θ.

Let us consider

$$\left.\begin{array}{rcl} \bar{x} & = & a(\theta, \mu)x, \\ \bar{\theta} & = & \theta + \omega, \end{array}\right\}$$

where a is a C^∞ function of θ and μ. We assume that $a(\cdot, \mu)$ has a zero of multiplicity 2 for $\mu = \mu_0$ at $\theta = \theta_0$,

$$\frac{\partial a}{\partial \mu}(\theta_0, \mu_0) \neq 0.$$

and that the number of zeros of $a(\cdot, \mu)$ increases from $\mu < \mu_0$ to $\mu > \mu_0$.

Theorem

Then, the Lyapunov exponent $\Lambda(\mu)$ is a C^0 function of μ such that:

1. *Λ is C^∞ when $\mu \neq \mu_0$.*
2. $$\lim_{\mu\to\mu_0^-} \Lambda'(\mu) = -\infty, \text{ and } \lim_{\mu\to\mu_0^+} \Lambda'(\mu) \text{ exists and is finite.}$$

Moreover, for $\mu \to \mu_0^-$ we have the asymptotic expression

$$\Lambda(\mu) = \Lambda(\mu_0) + A\sqrt{|\mu - \mu_0|} + O(|\mu - \mu_0|),$$

where $A > 0$.

Affine systems

$$\left.\begin{array}{rcl} \bar{x} & = & \alpha\, a(\theta)x + b(\theta), \\ \bar{\theta} & = & \theta + \omega, \end{array}\right\} \tag{2}$$

where a and b are C^r functions and α is a real positive parameter. It is clear that, for any invariant curve of (2), its linearized normal behaviour is described by

$$\left.\begin{array}{rcl} \bar{x} & = & \alpha\, a(\theta)x, \\ \bar{\theta} & = & \theta + \omega. \end{array}\right\} \tag{3}$$

In what follows, we will assume that (3) is not reducible.

The Lyapunov exponent is given by

$$\Lambda = \ln \alpha + \frac{1}{2\pi} \int_0^{2\pi} \ln |a(\theta)| \, d\theta.$$

If the integral above exists (and it is finite), then the Lyapunov exponent is negative for sufficiently small values of α, namely,

$$\alpha < \alpha_0 = \exp \left(-\frac{1}{2\pi} \int_0^{2\pi} \ln |a(\theta)| \, d\theta \right).$$

In particular this implies that, for $\alpha < \alpha_0$, *any* invariant curve of

$$\left. \begin{array}{rcl} \bar{x} & = & \alpha \, a(\theta)x + b(\theta), \\ \bar{\theta} & = & \theta + \omega, \end{array} \right\}$$

is attracting and, therefore, it must be unique.

Fractalization

As we are dealing with an affine system and the sup norm of a curve does not need to be bounded, we will say that a curve is fractalizing when its C^1 norm –taken on any closed nontrivial interval for θ– goes to infinity much faster than its C^0 norm, that is, when

$$\limsup_{\alpha \to \alpha_0} \frac{\|x'_\alpha\|_{I,\infty}}{\|x_\alpha\|_\infty} = +\infty,$$

where $\|\cdot\|_{I,\infty}$ denotes the sup norm on a nontrivial closed interval I.

Theorem

Assume that $a, b \in C^1(\mathbb{T}, \mathbb{R})$ *and that (3) is not reducible. Then,*

a) *If* $\limsup_{\alpha \to \alpha_0^-} \|x_\alpha\|_\infty < +\infty$,
and $b \in D_1$ *(D_1 is a suitable residual set), we have*

$$\limsup_{\alpha \to \alpha_0^-} \|x'_\alpha\|_{I,\infty} = +\infty,$$

for any nontrivial closed interval $I \subset \mathbb{T}$.

b) *If* $\limsup_{\alpha \to \alpha_0^-} \|x_\alpha\|_\infty = +\infty$,
then, for any nontrivial closed interval $I \subset \mathbb{T}$*, we have*

$$\limsup_{\alpha \to \alpha_0^-} \|x_\alpha\|_{I,\infty} = +\infty, \quad \text{and} \quad \limsup_{\alpha \to \alpha_0^-} \frac{\|x'_\alpha\|_{I,\infty}}{\|x_\alpha\|_\infty} = +\infty.$$

If a is a positive function with at least a zero (so that the skew product is not reducible), we have a better result.

Proposition

Assume, in (2), that $a, b \in C^1(\mathbb{T}, \mathbb{R})$, $a(\theta) \geq 0$ for all $\theta \in \mathbb{T}^1$ and there exists a value θ_0 such that $a(\theta_0) = 0$. We also assume that b never vanishes. Then,

a) *If $a, b \in C^r(\mathbb{T}, \mathbb{R})$, $r \geq 1$, then $x_\alpha \in C^r(\mathbb{T}, \mathbb{R})$ for $0 < \alpha < \alpha_0$.*

b) *For any nontrivial closed interval $I \subset \mathbb{T}$, we have*

$$\lim_{\alpha \to \alpha_0^-} \|x_\alpha\|_{I,\infty} = +\infty, \quad \textit{and} \quad \lim_{\alpha \to \alpha_0^-} \frac{\|x'_\alpha\|_{I,\infty}}{\|x_\alpha\|_\infty} = +\infty.$$

c) *For $\alpha > \alpha_0$, there is no $x \in C^0(\mathbb{T}, \mathbb{R})$ such that $x(\theta + \omega) = \alpha a(\theta) x(\theta) + b(\theta)$.*

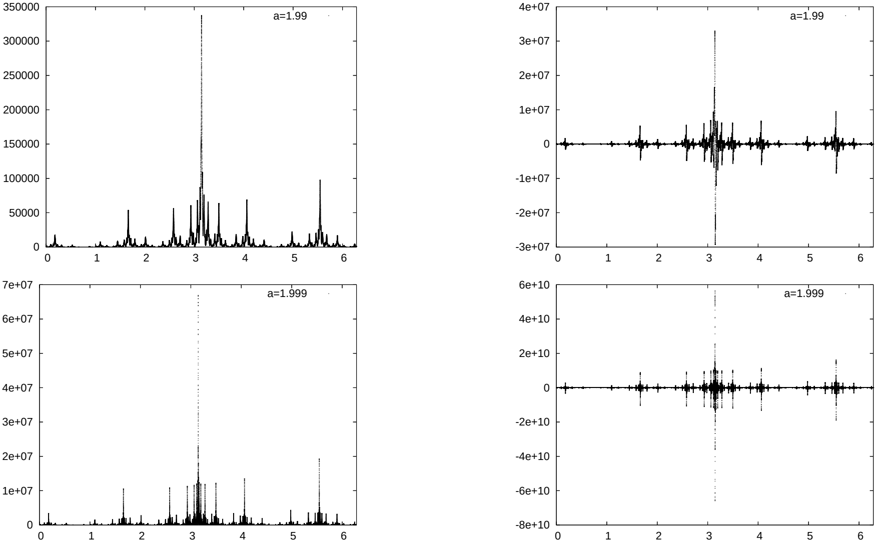
Some numerical examples

$$\left.\begin{array}{rcl} \bar{x} & = & \alpha\,(1+\cos\theta)x+1, \\ \bar{\theta} & = & \theta+\omega, \end{array}\right\}$$

where ω is the golden mean. We note that $1+\cos\theta \geq 0$ so we are in the hypotheses of the last proposition.

The Lyapunov exponent of the linear skew product is $\Lambda = \ln\alpha - \ln 2$ and, therefore, the critical value α_0 is 2.

Then, there exists a unique invariant attracting curve for $0<\alpha<2$, that undergoes a fractalization process when $\alpha \to 2^-$.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-3e+07
0 1 2 3 4 5 6
a=1.999
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3e+07
2e+07
1e+07
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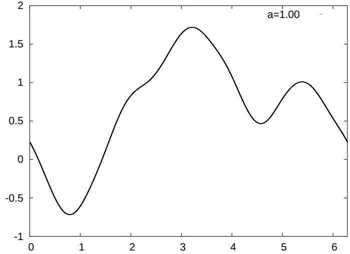
0 1 2 3 4 5 6
a=1.999
6e+10
4e+10
2e+10
0
-2e+10
-4e+10
-6e+10
-8e+10
0 1 2 3 4 5 6

Another example.

$$\left.\begin{array}{rcl} \bar{x} & = & \alpha \cos(\theta)\, x + 1, \\ \bar{\theta} & = & \theta + \omega, \end{array}\right\}$$

being α a positive parameter.
It is easy to see that its Lyapunov exponent is $\ln \alpha - \ln 2$.
If $\alpha < 2$, the Lyapunov exponent is negative. Therefore, we must have a unique and global attracting set.
Next slides show the attractor for several values $\alpha < 2$.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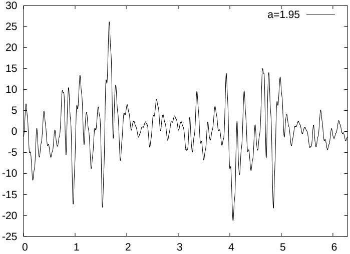
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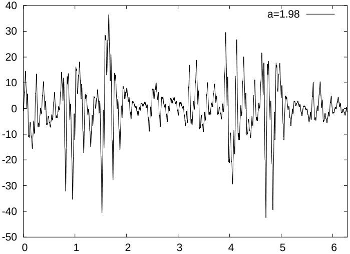
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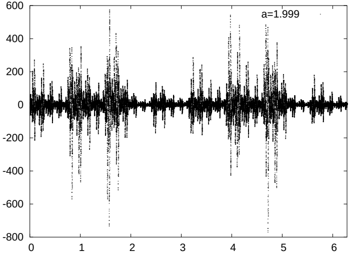
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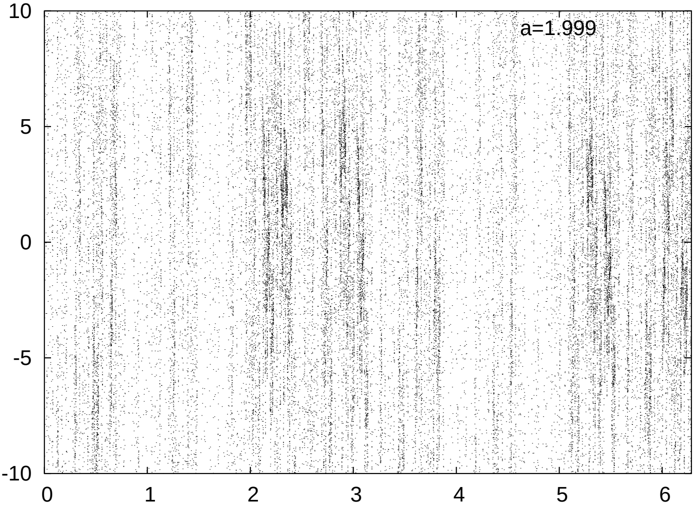
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-5
-10
0
1
2
3
4
5
6

Affine skew products of the plane

Some skew products of the plane

We have shown that, in some situations, the lack of reducibility implies the existence of "weird" behaviours.

Here we will look at fractalization processes in (a bit) higher dimensions.

Let us start considering the following situation:

$$\begin{aligned} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} &= \mu \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c \\ 0 \end{pmatrix}, \\ \bar{\theta} &= \theta + \omega, \end{aligned}$$

where c and μ are real parameters.

It is easy to prove that

1. if $|\mu| < 1$ the map has an attracting invariant curve,
2. if $|\mu| > 1$ the map has a repelling curve,
3. if $|\mu| = 1$ the map has no invariant curve.

Let us show the behaviour of this system by means of a numerical experiment.

In what follows, let us fix $c = 1$.

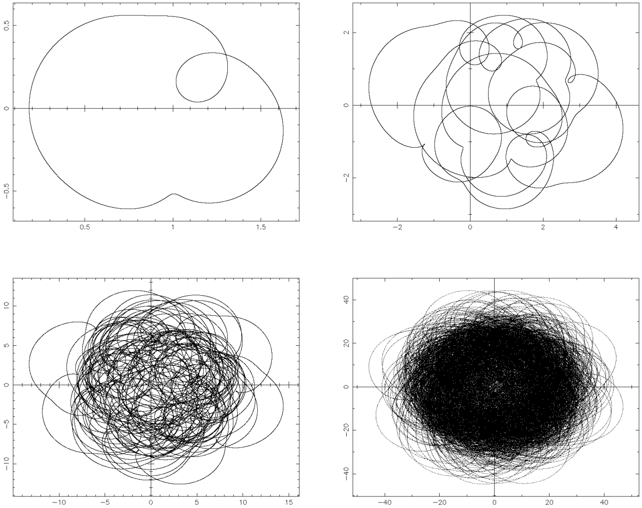

Figure: Attracting curve for $\mu = 0.5$, $\mu = 0.9$, $\mu = 0.99$ and $\mu = 0.999$

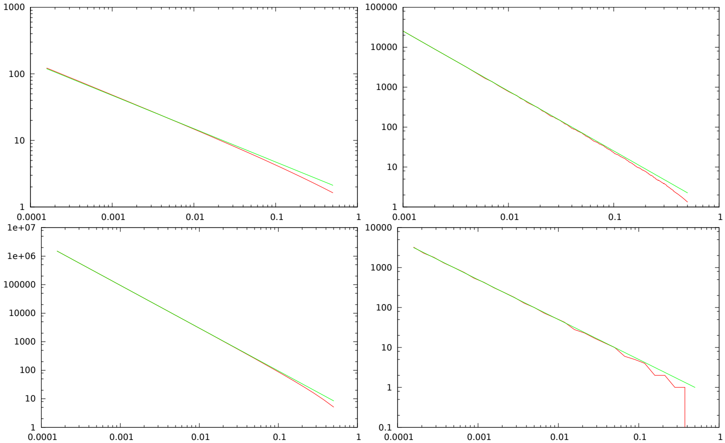

Figure: Computations with $c = 1$. **Top:** On the left, fitting between $\|z_\mu\|_\infty$ and $2(1-\mu)^{-1/2}$. On the right, fitting between $\|z'_\mu\|_\infty$ and $\frac{4}{5}(1-\mu)^{-3/2}$. **Bottom:** On the left, fitting between the length of z_μ and $3(1-\mu)^{-3/2}$. On the right, fitting between wind$(z_\mu, 0)$ and $\frac{1}{2}(1-\mu)^{-1}$.

Note that the linear dynamics along these curves is given by

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \mu \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$
$$\bar{\theta} = \theta + \omega$$

and that this linear system is not reducible.

Let us use complex notation: if $z = x + y\mathrm{i}$, the previous map can be written as

$$\tilde{z} = \mu e^{\theta \mathrm{i}} z + c,$$
$$\tilde{\theta} = \theta + \omega.$$

So, let us start focusing on linear invertible maps of the form

$$\begin{array}{rccc} F_{a(\theta)}: & \mathbb{T}\times\mathbb{C} & \longrightarrow & \mathbb{T}\times\mathbb{C} \\ & (\theta, z) & \longmapsto & (\theta+\omega, a(\theta)z), \end{array}$$

where, for all $\theta \in \mathbb{T}$, the value $a(\theta) \in \mathbb{C}$ is different from zero.

Definition

A number ω is called *Diophantine* of type (γ, τ) for $\gamma > 0$ and $\tau \geq 2$, if

$$\left|\omega - \frac{p}{q}\right| > \frac{\gamma}{|q|^{\tau}}$$

for all $\frac{p}{q} \in \mathbb{Q}$. We denote by $\mathcal{D}_{\gamma,\tau}$ the set of Diophantine numbers.

Moreover, we assume than the map $\theta \mapsto a(\theta)$ is of class C^r ($r \geq 1$) and that $\omega \in \mathcal{D}_{\gamma,\tau}$. We are interested in classifying these linear skew-products

Let us denote by $\mathrm{wind}(a(\theta), 0)$ the winding number of the closed curve $a(\theta)$ with respect to the point $z = 0$.

Definition (Topological and linear conjugacy as skew products)

Two linear skew-products F_a and F_b are topologically conjugate as skew products if there exists a change of coordinates of the form

$$\mathcal{H}(\theta, z) = (\theta + \nu, H(\theta, z))$$

where $\nu \in \mathbb{T}$ and, for each θ, $H(\theta, \cdot)$ is a homeomorphism of the plane verifying $H(\theta, 0) = 0$ and such that

$$\mathcal{H}^{-1} \circ F_a \circ \mathcal{H} = F_b.$$

When $H(\theta, z)$ can be chosen to be linear w.r.t. z, i.e. $H(\theta, z) = c(\theta)z$, with $c(\theta)$ continuous and different from zero for all θ, then F_a and F_b are said to be linearly conjugate as skew products up to an angle translation. If $\nu = 0$ we say that F_a and F_b are linearly conjugate as skew products.

Definition

A topological conjugacy as above is isotopic to the identity if $H(\theta, \cdot)$ is isotopic to the identity for each $\theta \in \mathbb{T}$.

Note that linear conjugacies are always isotopic to the identity.

Definition (Reducibility)

A linear skew product

$$\left.\begin{array}{rcl} \theta & \mapsto & \theta + \omega, \\ z & \mapsto & a(\theta) z, \end{array}\right\}$$

is said to be *reducible* iff there exists a linear change of variables, $(\theta, z) = (\theta, e(\theta) u)$ such that the transformed system becomes

$$\left.\begin{array}{rcl} \theta & \mapsto & \theta + \omega, \\ u & \mapsto & bu, \end{array}\right\}$$

where $b = e(\theta + \omega)^{-1} a(\theta) e(\theta)$ does not depend on θ.

Proposition

Let $\omega \in \mathcal{D}_{\gamma,\tau}$. Then there exists $r = r(\tau)$ such that if $a(\theta)$ and $b(\theta)$ are of class C^r then the following equivalence holds: $F_{a(\theta)}$ and $F_{b(\theta)}$ are linearly conjugated if and only if the following two conditions are satisfied:

(a) $\mathrm{wind}(a(\theta), 0) = \mathrm{wind}(b(\theta), 0)$.

(b) *There exists $m \in \mathbb{Z}$ and a branch of the logarithm such that*

$$\int_{\mathbb{T}} \log\left(e^{-im\omega} \frac{a(\theta)}{b(\theta)} \right) \, d\theta = 0.$$

Moreover, if such m exists, it is unique and the linear change of coordinates $H_\theta(z) = c(\theta)z$ satisfies that $\mathrm{wind}(c(\theta), 0) = m$.

Proposition (Normal form)

Assume $\omega \in \mathcal{D}_{\gamma,\tau}$, $a(\theta)$ is $C^{r(\tau)}$ and $\mathrm{wind}(a(\theta), 0) = n$. Then, for any $m \in \mathbb{Z}$, there exists a linear change, of winding number $-m$, which conjugates $F_{a(\theta)}$ to

$$F_{b(m,\theta)}(\theta, z) = (\theta + \omega, be^{im\omega}e^{in\theta}z)$$

where $b = |b|e^{i\rho} \in \mathbb{C}$ satisfies

$$|b| = \exp\left(\frac{1}{2\pi}\int_{\mathbb{T}} \log|a(\theta)|\, d\theta\right), \quad \rho = \frac{1}{2\pi}\mathrm{Im}\int_{\mathbb{T}} \log(a(\theta)e^{-in\theta})\, d\theta,$$

for any determination of the logarithm. Moreover, two such systems $(\theta + \omega, b_1 e^{in\theta}z)$ and $(\theta + \omega, b_2 e^{in\theta}z)$, with $b_1, b_2 \in \mathbb{C}$ are linearly conjugate if and only if $b_1 = b_2 e^{im\omega}$ for some $m \in \mathbb{Z}$.

Corollary

Assume $\omega \in \mathcal{D}_{\gamma,\tau}$, $a(\theta)$ *is* $C^{r(\tau)}$. *If* $\operatorname{wind}(a(\theta), 0) = 0$, *then the system is reducible. Moreover, the system is reducible to a system of the form* $(\theta + \omega, bz)$ *with* $b \in \mathbb{R}$, *if and only if there exists* $m \in \mathbb{Z}$ *and a branch of the argument such that*

$$\int_{\mathbb{T}} \arg(a(\theta))d\theta - m\omega = 0.$$

In such case, the change has winding number equal to $-m$.

Corollary

Assume $\omega \in \mathcal{D}_{\gamma,\tau}$, $a(\theta)$ *is* $C^r(\tau)$ *and* $\mathrm{wind}(a(\theta), 0) = n \neq 0$. *Then, there exists a unique* $b \in \mathbb{R}$ *such that* $F_{a(\theta)}$ *is linearly conjugate to*

$$F_b(\theta, z) = (\theta + \omega, be^{in\theta} z).$$

As before, the precise value of b *is*

$$b = \exp\left(\frac{1}{2\pi}\int_{\mathbb{T}} \log|a(\theta)|\, d\theta\right).$$

Topological conjugacy classes

We recall that if two linear skew products $F_{a(\theta)}$ and $F_{b(\theta)}$ are topologically conjugate then there exists a constant $\nu \in \mathbb{T}$ and a homeomorphism $H : \mathbb{T} \times \mathbb{C} \to \mathbb{C}$ such that

$$H(\theta + \omega, a(\theta)z) = b(\theta + \nu)H(\theta, z), \quad \forall \theta \in \mathbb{T}, \; \forall z \in \mathbb{C}.$$

Proposition

If two linear skew products $F_{a(\theta)}$ and $F_{b(\theta)}$ are topologically conjugate then $\mathrm{wind}(a(\theta), 0) = \mathrm{wind}(b(\theta), 0)$.

Theorem

Assume that $\omega \in \mathcal{D}_{\gamma,\tau}$, and a is an invertible $C^{r(\tau)}$ function. Then the skew-product is topologically conjugate to one of the following:

a) *If* $\mathrm{wind}(a(\theta), 0) = 0$ *and the Lyapunov exponent is negative,*

$$\left.\begin{array}{rcl} \tilde{\theta} & = & \theta + \omega, \\ \tilde{z} & = & \frac{1}{2} z, \end{array}\right\}$$

b) *If* $\mathrm{wind}(a(\theta), 0) = 0$ *and the Lyapunov exponent is positive,*

$$\left.\begin{array}{rcl} \tilde{\theta} & = & \theta + \omega, \\ \tilde{z} & = & 2z, \end{array}\right\}$$

c) *If* $\mathrm{wind}(a(\theta), 0) = 0$ *and the Lyapunov exponent is zero,*

$$\left.\begin{array}{rcl} \tilde{\theta} & = & \theta + \omega, \\ \tilde{z} & = & e^{i\rho} z, \end{array}\right\}$$

Theorem (cont.)

d) *If* $\mathrm{wind}(a(\theta), 0) = n \neq 0$ *and the Lyapunov exponent is negative,*

$$\left.\begin{array}{rcl} \tilde{\theta} & = & \theta + \omega, \\ \tilde{z} & = & \frac{1}{2} e^{in\theta} z, \end{array}\right\}$$

e) *If* $\mathrm{wind}(a(\theta), 0) = n \neq 0$ *and the Lyapunov exponent is positive,*

$$\left.\begin{array}{rcl} \tilde{\theta} & = & \theta + \omega, \\ \tilde{z} & = & 2 e^{in\theta} z, \end{array}\right\}$$

f) *If* $\mathrm{wind}(a(\theta), 0) = n \neq 0$ *and the Lyapunov exponent is zero,*

$$\left.\begin{array}{rcl} \tilde{\theta} & = & \theta + \omega, \\ \tilde{z} & = & e^{in\theta} z, \end{array}\right\}$$

Lyapunov exponents

Definition

Let us consider a linear quasi-periodic skew product given by $a \in \mathcal{C}^r(\mathbb{T}, \mathbb{C})$, $r \geq 0$. Fix $\theta \in \mathbb{T}$, we define the Lyapunov exponent at θ of the skew-product as

$$\lambda(\theta) = \limsup_{n \to \infty} \frac{1}{n} \ln \left| \prod_{j=0}^{n-1} a(\theta + j\omega) \right|.$$

We also define the Lyapunov exponent of the skew-product as

$$\Lambda = \frac{1}{2\pi} \int_{\mathbb{T}} \ln |a(\theta)| \, d\theta.$$

Theorem

Let us consider a one-parametric family of quasi-periodic cocycles

$$\left.\begin{array}{rcl} \tilde{z} & = & a(\theta,\mu)z, \\ \tilde{\theta} & = & \theta+\omega, \end{array}\right\}$$

where ω is Diophantine, μ belongs to an open nonempty interval $I \subset \mathbb{R}$ and $a \in \mathcal{C}^\infty(\mathbb{T}\times I, \mathbb{C})$. We assume that

1. *There exists a unique pair (θ_0,μ_0) such that $a(\theta_0,\mu_0)=0$.*
2. $$\frac{\partial a}{\partial \theta}(\theta_0,\mu_0) \neq 0, \qquad \mathrm{Im}\,\frac{\frac{\partial a}{\partial \mu}(\theta_0,\mu_0)}{\frac{\partial a}{\partial \theta}(\theta_0,\mu_0)} \neq 0,$$

Theorem (cont.)

then, the Lyapunov exponent $\Lambda(\mu)$ is a continuous function of μ such that

1. *It is $\mathcal{C}^\infty$ at any $\mu \neq \mu_0$.*
2. *It is $\mathcal{C}^0$ at $\mu = \mu_0$ and there exist constants A^+ and A^-, for which, when $\mu \to \mu_0$, the following expression holds:*

$$\Lambda(\mu) = \Lambda(\mu_0) + A^{\pm}(\mu - \mu_0) + \mathcal{O}(|\mu - \mu_0|^2)$$

   where A^+ is used when $\mu > \mu_0$ and A^- when $\mu < \mu_0$. The values A^+ and A^- never coincide.

Fractalization

Definition (Fractalization process)

Consider a curve $z_\mu \in \mathcal{C}^r(\mathbb{T}, \mathbb{C})$, $r \geq 1$, depending on a real parameter μ. We say that the curve undergoes a fractalization process if there exists some critical value $\mu^\star$ such that

$$\limsup_{\mu \to \mu^\star} \frac{\|z'_\mu\|_{I,\infty}}{\|z_\mu\|_\infty} = \infty,$$

where $\|\cdot\|_{I,\infty}$ denotes the sup norm on a nontrivial closed interval I.

Definition (Wild winding process)

Let $z_\mu \in \mathcal{C}^r(\mathbb{T}, \mathbb{C})$, $r \geq 1$ and S any subset of $\mathbb{C}$. If for any $s \in S$ there exists a monotonically increasing sequence $\{\mu_j\}_{j\in\mathbb{N}}$ such that

1. $\lim_{j\to\infty} \mu_j = \mu^\star$,
2. for each j, $z_{\mu_j}(\theta) \neq s$ for all $\theta \in \mathbb{T}$
3. and $\lim_{j\to\infty} |\operatorname{wind}(z_{\mu_j}, s)| = \infty$,

then we say that z_μ is undergoing a wild winding process on S from below when $\mu \to \mu^\star$.

Theorem

Assume that ω is of constant type. Consider

$$\tilde{z} = \mu e^{\mathrm{i}\,\theta} z + c, \qquad \tilde{\theta} = \theta + \omega,$$

where $z \in \mathbb{C}$, $\theta \in \mathbb{T}$ and $\mu \in \mathbb{R}$ is a parameter. Then:

1. *This system has a unique invariant curve z_μ for each $\mu \neq 1$. The invariant curve is attracting if $\mu < 1$ and repelling if $\mu > 1$.*
2. *The invariant curve undergoes a fractalization process when $\mu \to 1$. Moreover, if $\mu \to 1$,*

$$\frac{\|z'_\mu\|_\infty}{\|z_\mu\|_\infty} = \mathcal{O}(1-\mu)^{-1}.$$

3. *The invariant curve undergoes a wild winding process on $\mathbb{C}$ when $\mu \to 1$. Moreover, if $\mu \to 1$,*

$$\mathrm{wind}(z_\mu, s) = \mathcal{O}(1-\mu)^{-1} \qquad \textit{for each } s \in \mathbb{C}.$$

The proof is based in several facts. The first one is that, due to the simplicity of the model, it is not difficult to find a Fourier series for the invariant curve. For instance, if $\mu < 1$, the invariant curve is given by

$$z_\mu(\theta) = c \sum_{k=0}^{\infty} \mu^k e^{-\mathrm{i}\,\frac{k(k+1)}{2}\omega} e^{\mathrm{i}\,k\theta}.$$

(a similar expression can be derived if $\mu > 1$). Note that this series defines an analytic function.

The second (and key) fact is found in a paper by Hardy & Littlewood: *Some problems of diophantine approximation*, Acta Mathematica 37:1, pp. 193-239 (1914).

The results in this paper show that, for ω of constant type, the value of the series grows to infinity when μ goes to 1. More concretely,

$$\begin{aligned} z_\mu(\theta) &= \mathcal{O}(1-\mu)^{-1/2}, \\ z'_\mu(\theta) &= \mathcal{O}(1-\mu)^{-3/2}. \end{aligned}$$

The last fact in the proof is based on the argument principle.

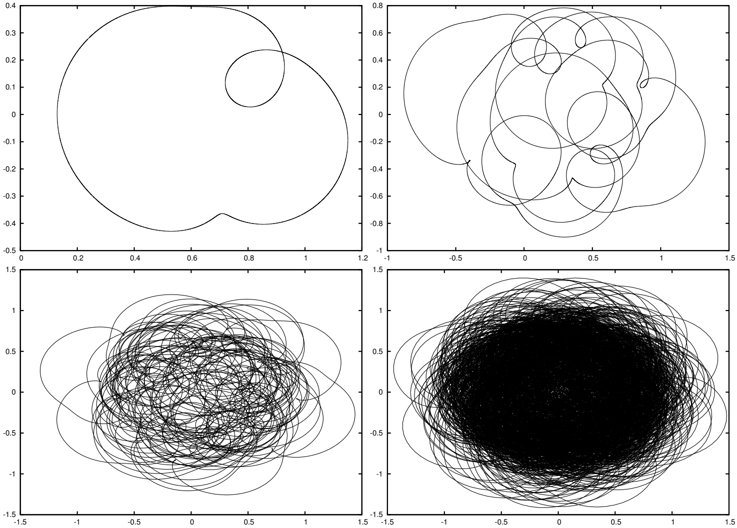

Figure: A winding process on $D_0(1)$ is displayed. Invariant curve with $n = 1$ and $c = \sqrt{1-\mu}$. Plots for $\mu = 0.5$, $\mu = 0.9$, $\mu = 0.99$ and $\mu = 0.999$.

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