STRAIGHTENING THE SQUARE

Conformal/affine geometry, flat connections and the Schwarz-Christoffel formula

What if you live

in a flat world

where size is not well-defined?

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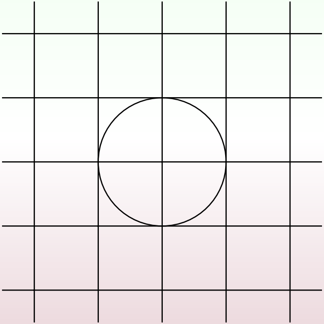
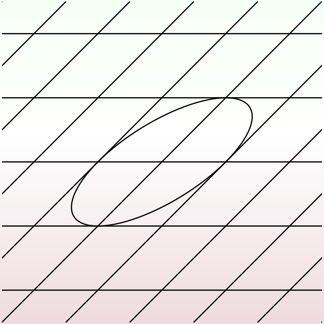
Linear maps in the plane

do not preserve circles





Beltrami derivative

• $\mathbb{C}$ = the complex numbers = a Euclidean plane $\equiv \mathbb{R}^2$

• f: a map from (a domain of) the plane to (a domain of) the plane. In this whole talk, we assume f orientation preserving, i.e. $\det(df) > 0$.

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The *Beltrami derivative* it is a complex number $Bf(z)$ that encodes faithfully the flatness and orientation of the pre-image of a circle by df.

– $Bf(z)$ can be any complex number of modulus < 1

– $Bf = 0$ iff df is a similitude

Ellipse fields

An *ellipse field* is the data of (infinitesimal) ellipses attached to each point of a domain. Their size is not relevant, only their direction and flatness, encoded by a complex number $\mu(z)$ with the same convention as for Bf.

Straightening the ellipse field is solving the differential equation $Bf = \mu$, i.e. finding a deformation f of the plane whose differential sends all the ellipses to circles.

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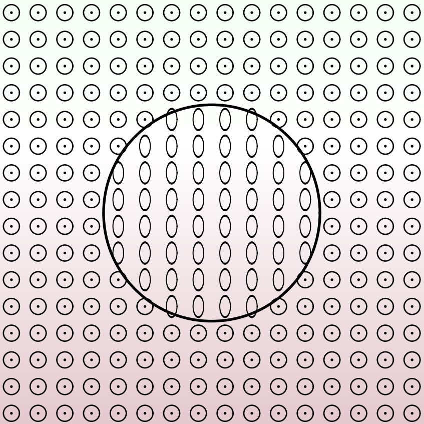
Uses, existence, formula?

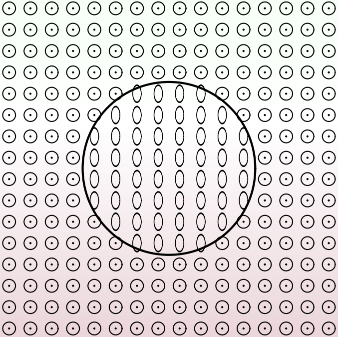
The disk

Setting: an ellipse field on the plane that is constant in the unit disk ($\mu = a$), and circles outside ($\mu = 0$). Problem: find the straightening.



The disk

an amazing coincidence



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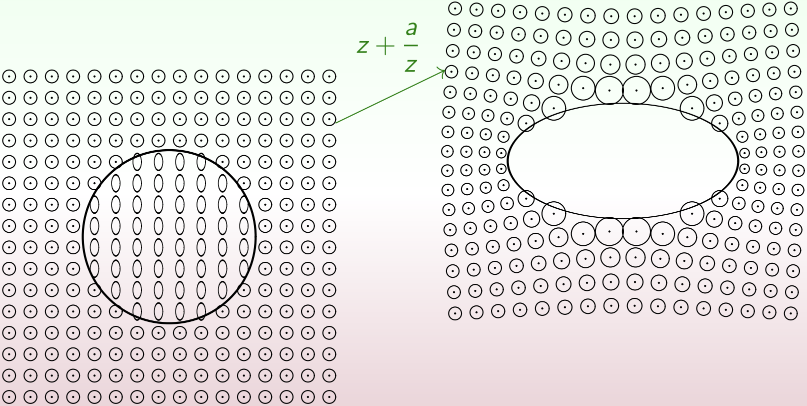

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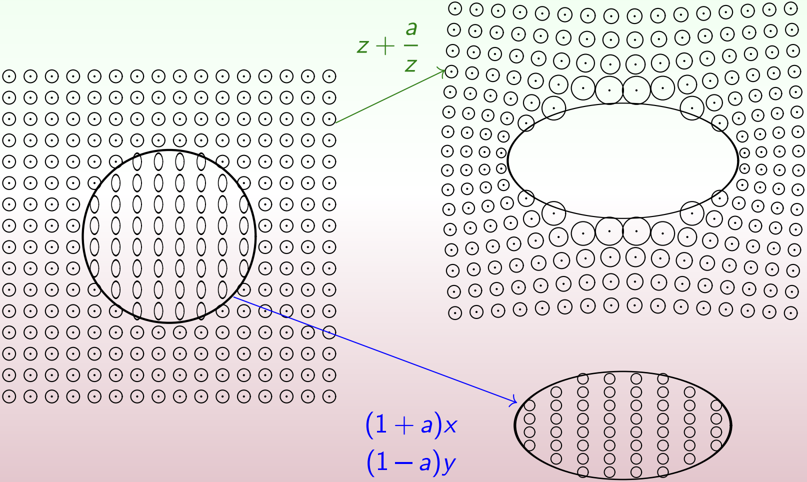

The disk

an amazing coincidence (a conspiracy?)

Recall that if $z = x + iy$ then $\overline{z} = x - iy$ and $|z|^2 = x^2 + y^2 = z\overline{z}$, hence

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In particular for z on the unit circle:

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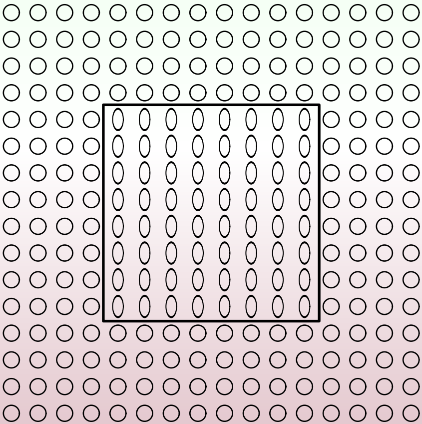

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An anecdote

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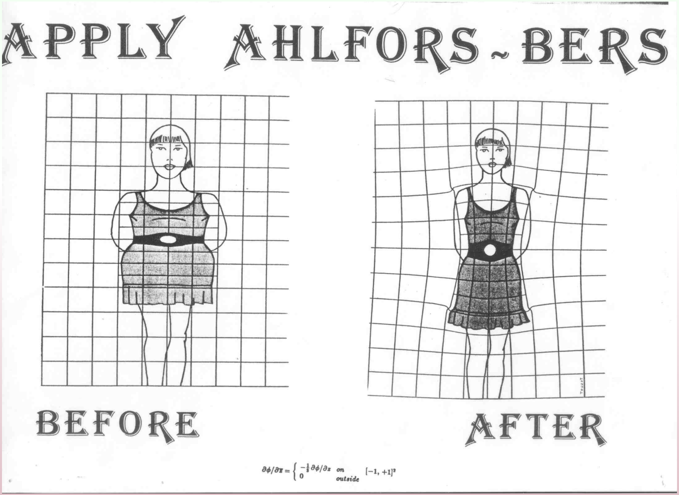

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$$\partial\phi/\partial\bar{z} = \begin{cases} -\frac{1}{3}\,\partial\phi/\partial z & \text{on} \quad [-1,+1]^2 \\ 0 & \text{outside} \end{cases}$$

What does it look like on Douady?

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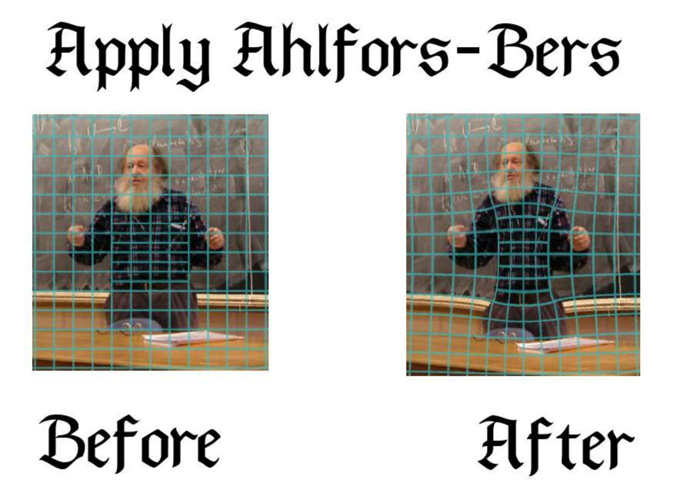

The square

a naive attempt

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The square

An obstruction to naiveness from potential theory

Because conformal maps must preserve the solutions of Laplace's equation $\Delta V = 0$, energy considerations imply that a side of the rectangle cannot be mapped to a curve with a too small diameter.

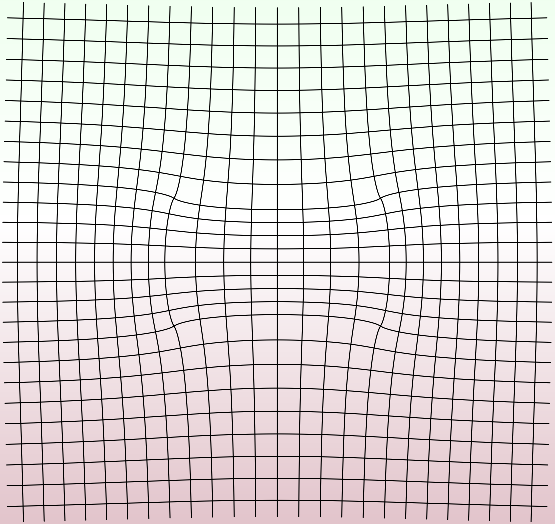
The solution f of the Beltrami equation is harmonic for a modified Laplacian:

$$\widetilde{\Delta} f = 0.$$

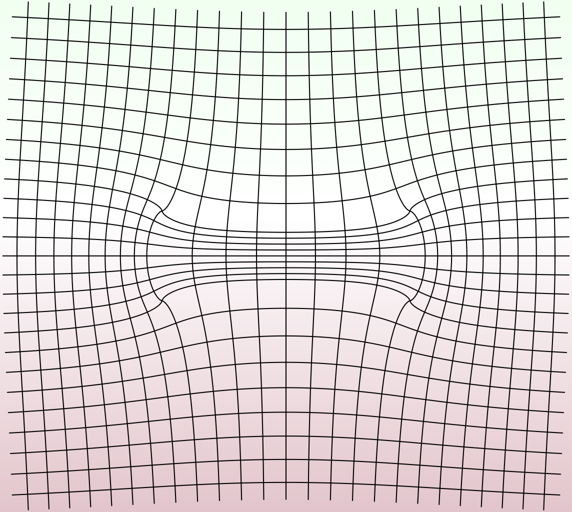
There are several well-studied schemes to solve this kind of equations numerically. To obtain the following set of pictures, I worked on a grid, used a discrete modified laplacian, and approximated a solution using the *Jacobi relaxation method*, an iterative method that converges rapidly.

Numerically solving a modified Laplacian

$K = 2$

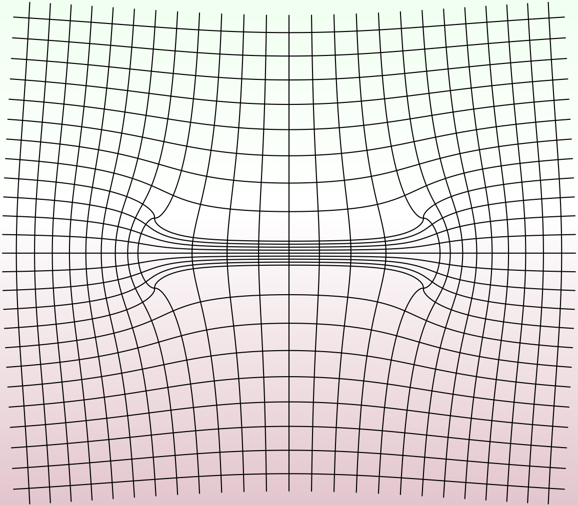


$K = 5$



Numerically solving a modified Laplacian

$K = 10$



Numerically solving a modified Laplacian

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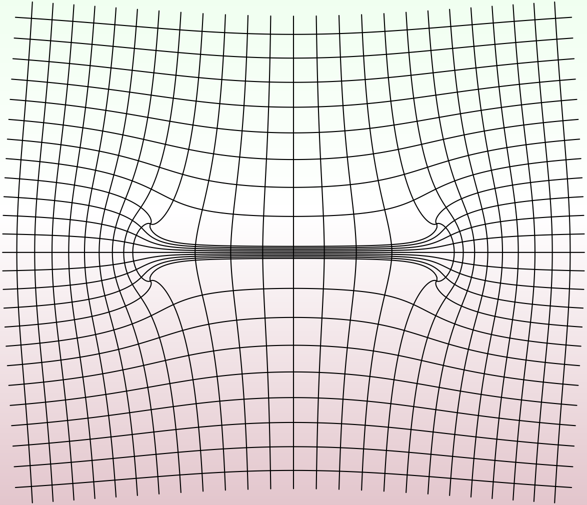
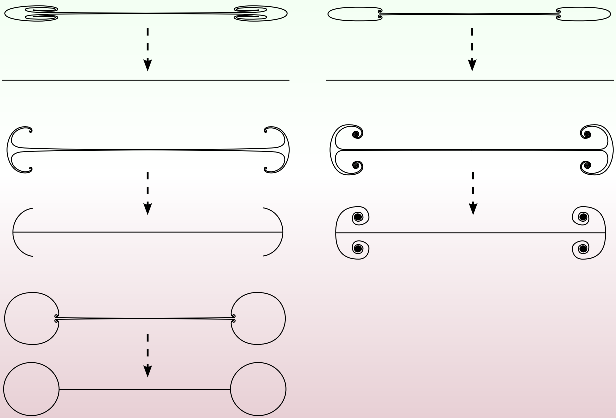
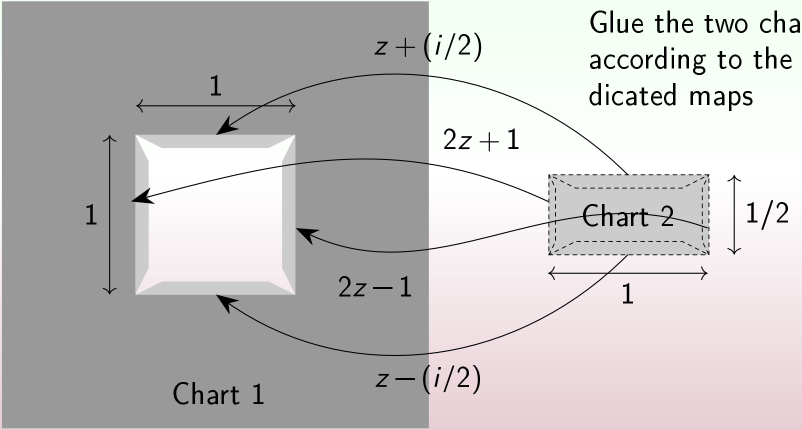


Limit as $K \to +\infty$

Guesses for the square



The square
Reformulation



Glue the two charts according to the indicated maps

The square

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The change of coordinates are similitudes, so we work with a more rigid category of geometrical object, *similarity surfaces*, with interesting properties like...

...a locally trivial parallel transport.

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The square

How do ones living there *see* their world?

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Uniformization theorem

Theorem: (Poincaré, Koebe) *A Riemann surface that is homeomorphic to a sphere is necessarily conformally equivalent to the Euclidean sphere.*

In our case, we can complete our gluing by adding 5 points, one at infinity, four at the corners, and 5 Riemann charts near these points.

Completing the Riemann surface

1. Near ∞, the map $z \mapsto 1/z$ gives a local chart (exactly like the *Riemann sphere*).

2. Near a corner, we can glue one side of the rectangle to one side of the square and are left with the following local picture: a slit plane where one side of the slit is glued to the other side by a homothety of ratio K.

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Then the map

$$z \mapsto z^{\alpha}, \qquad \alpha = \frac{2\pi i}{2\pi i \pm \log K}$$

is a local chart: in particular it glues each side of slit exactly according to the required homothety.

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A cultural remark

M.C. Escher's lithography: Print Gallery (1956)

A solution via uniformization

The Euclidean sphere minus one point is conformally equivalent to the plane (stereographic projection).

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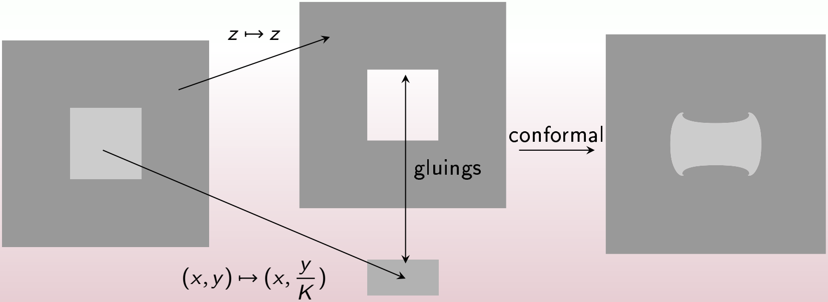

Explicit uniformization?

But usually finding the explicit uniformization of abstract Riemann surfaces is a very hard problem, so what helps us here?

The global chart $\mathbb{C} \setminus \{z_1, \ldots, z_4\}$ is a Riemann chart but not a sim-chart. The change of coordinates from this chart to the sim-charts are *holomorphic* functions $\phi : U \to \mathbb{C}$ with $U \subset \mathbb{C} \setminus \{z_1, \ldots, z_4\}$. For two such sim-charts, ϕ_1, ϕ_2, then on $U_1 \cap U_2$ they satisfy (locally)

$$\phi_1 = a\phi_2 + b$$

for some constants a, b. Hence

$$\frac{\phi_2''}{\phi_2'} = \frac{\phi_1''}{\phi_1'} \ .$$

It follows that there exists a global holomorphic function

$$\eta : \mathbb{C} \setminus \{z_1, \dots, z_4\} \to \mathbb{C}$$

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Note: (differential geometry viewpoint) the function η is the expression* of a holomorphic and locally flat *connection*.

(*) a.k.a. a Christoffel symbol.

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Analyzing η at the singularities

Change of variable for the connection: if one expresses η in two Riemann charts C_1 and C_2 with change of coordinates ψ between them, then the expressions η_1 and η_2 in the respective charts are related by:

$$\eta_2 = \psi' \times \eta_1 \circ \psi + \frac{\psi''}{\psi'} \ . \tag{1}$$

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For the slit plane model, recall the gluing $z \mapsto z^\alpha$ with $\alpha = \frac{2\pi i}{2\pi i \pm \log K}$.

Then $\phi = z^{1/\alpha}$ hence $\phi''/\phi' = \frac{\frac{1}{\alpha}-1}{z}$:

$$\eta_1 = \frac{\log K}{2\pi i} \cdot \frac{1}{z}.$$

By (1), η_2 has a simple pole at z_i and its polar part is $\frac{\log K}{2\pi i} \cdot \frac{1}{z-z_i}$.

Solution

As a consequence:

- η has a simple pole at z_k with residue $\log(K)/2\pi i$.
- $\eta \longrightarrow 0$ when $z \longrightarrow \infty$

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$$\eta = \frac{\log K}{2\pi i} \cdot \left(\frac{-1}{z - z_1} + \frac{1}{z - z_2} + \frac{-1}{z - z_3} + \frac{1}{z - z_4} \right)$$

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Now solving $\phi''/\phi' = \eta$ gives:

$$\phi = b + a \int \left(\frac{z - z_2}{z - z_4} \cdot \frac{z - z_1}{z - z_3} \right)^{\frac{\log K}{2\pi i}} dz.$$

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The Schwarz-Christoffel formula

The formula we found

$$a+b\int\left(\frac{z-z_2}{z-z_4}\cdot\frac{z-z_1}{z-z_3}\right)^{\frac{\log K}{2\pi i}}dz$$

is an analogue of the Schwarz-Christoffel formula that gives an expression of the conformal map from the upper half plane to any polygon in the plane: for an n-gon with angles $\alpha_k \in (0,2\pi)$, there exists real numbers $x_1,\ldots,x_n$ such that

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with $\beta_k = 1-\frac{\alpha_k}{\pi}$.

The x_i are mapped to the vertices of the polygon. They can be hard to determine: each depends on all the angles and the length of all sides of the polygon. This is called the *parameter problem*.

Parameter problem

Similarly in our question we face a parameter problem: finding the values of $z_1,\ldots,z_4$. Using the symmetries, this reduces to finding the shape ratio K' of the rectangle $z_1,\ldots,z_4$ as a function of K.

(Note: $K' \neq K$.)

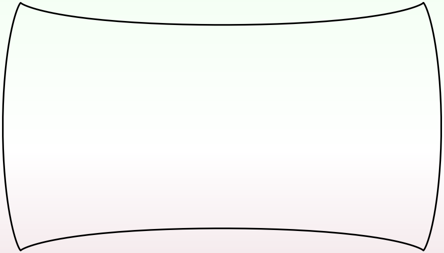
We thus have a one (real) parameter equation of unknown K':

$$\text{(E)} \qquad \frac{\int_{[z_1,z_2]}\omega}{\int_{[z_4,z_1]}\omega} = iK.$$

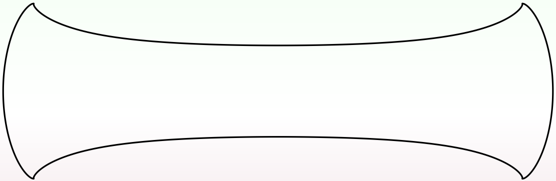
with $\omega = \left(\frac{z-z_2}{z-z_4}\cdot\frac{z-z_1}{z-z_3}\right)^{\frac{\log K}{2\pi i}} dz.$

I resorted to solve (E) *numerically* for each explicit value of K (this is not too hard).

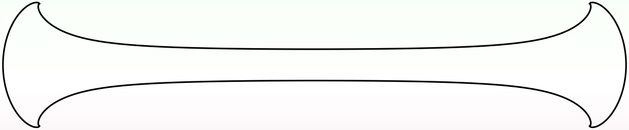
$K = 2$



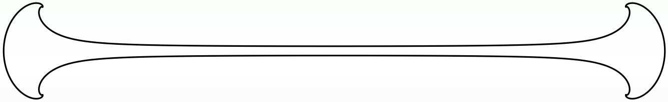
$K = 5$



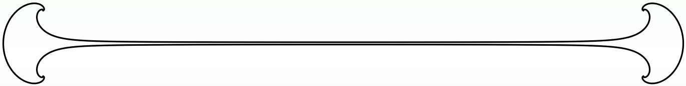
$K = 15$



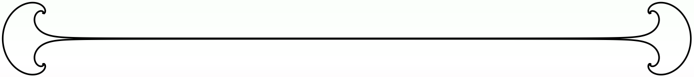
$K = 50$



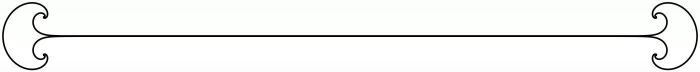
$K = 200$



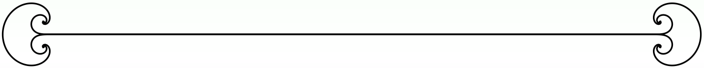
$K = 1000$



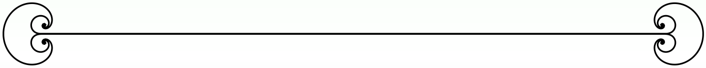
$K = 10^4$



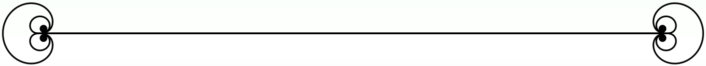
$K = 10^6$



$K = 10^9$



$K = 10^{20}$



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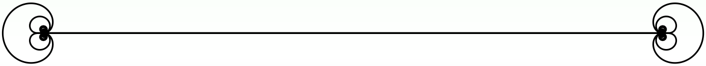

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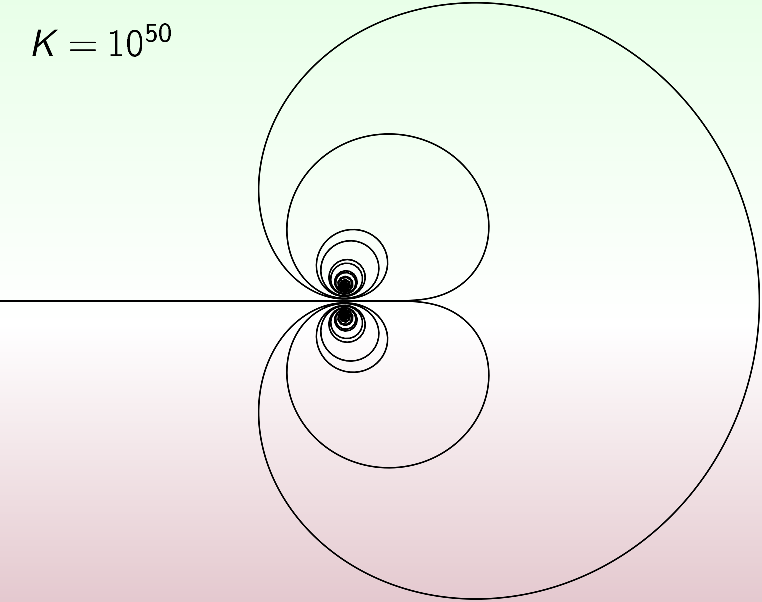
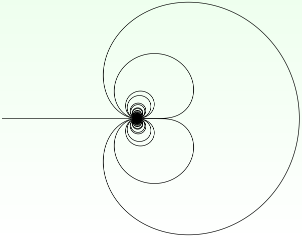


The limit



As $K \longrightarrow +\infty$ we see a limit shape and can prove

$$\eta_K \longrightarrow \eta_\infty = \frac{\sigma_0}{(z-x_0)^2} - \frac{\sigma_0}{(z+x_0)^2}$$

This limit shape also has an interpretation in terms of similarity surfaces:

`Click here to run applet`

The limit