Irreductibility in Holomorphic Dynamics

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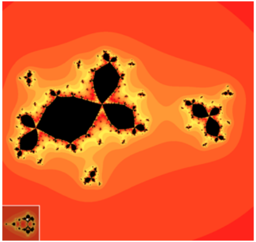
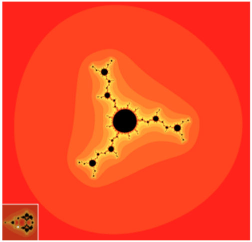
joint work with Adam Epstein and Sarah Koch

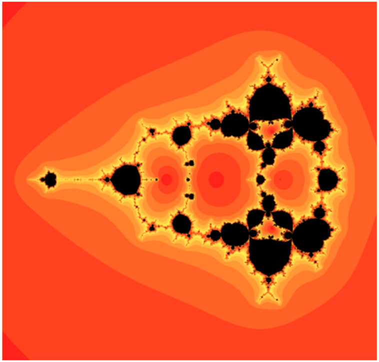
Special curves

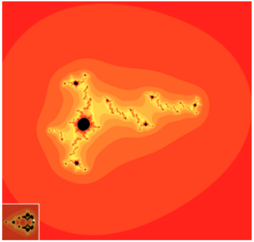
- $\mathscr{P}_3$ is the dynamical moduli space of cubic polynomials modulo affine conjugacy.
- $\mathscr{M}_2$ is the dynamical moduli space of quadratic rational maps modulo conjugacy by Möbius transformations.
- $\mathscr{S}_{k,n} \subset \mathscr{P}_3$ (resp. $\mathscr{V}_{k,n} \subset \mathscr{M}_2$) is the curve of conjugacy classes of cubic polynomials (resp. quadratic rational maps) having a critical point preperiodic to a cycle of period n with preperiod k.

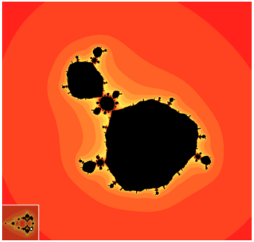
$\mathscr{S}_{0,3}$

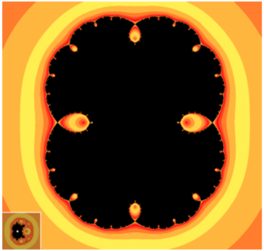


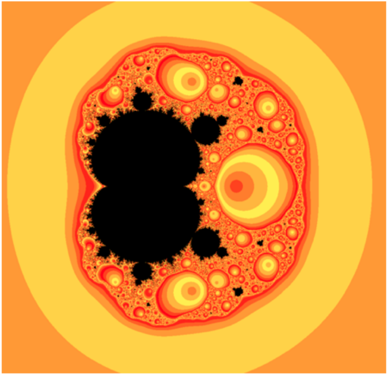


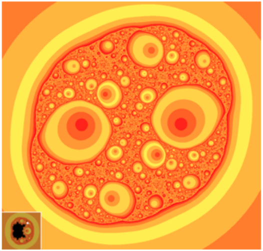


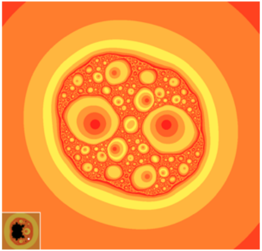


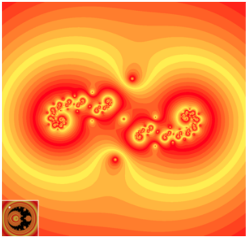


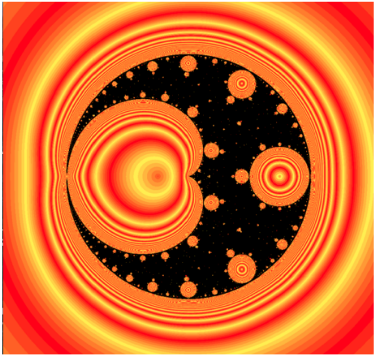


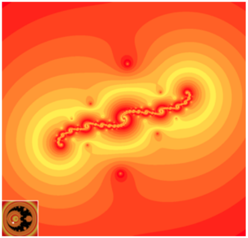












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Conjecture (Milnor)

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Theorem (B.-Epstein-Koch)

For all $k \geq 0$, the curve $\mathscr{S}_{k,1}$ is irreducible.

Theorem (B.-Epstein-Koch)

For all $k \geq 2$, the curve $\mathscr{V}_{k,1}$ is irreducible.

Equation of $\mathscr{S}_{k,1}$

- $F_{a,b}(z) = z^3 - 3a^2z + 2a^3 + b, \quad (a,b) \in \mathbb{C}^2$.
- $\mathscr{P}_3$ is obtained by identifying (a,b) with $(-a,-b)$.

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- $P_k := F_{a,b}^{\circ k}(a)$:

$$P_0 = a, \quad P_1 = b \quad \text{and} \quad P_{k+1} = P_k^3 - 3a^2P_k + 2a^3 + b.$$

- $F_{a,b}(z) - F_{a,b}(w) = (z-w)(z^2 + zw + w^2 - 3a^2)$.

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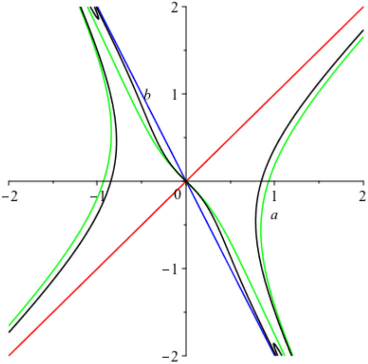
Proposition

The polynomial R_k is irreducible.

Equation of $\mathscr{S}_{k,1}$

- $R_1 = 2a + b$
- $R_2 = (2a+b)^2(b-a)^3 - 3b(2a+b)(a-b) + 3(a+b)$.
- $R_3 = (2a+b)^6(b-a)^{11} + \cdots + 3(a+b)$.



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Proof: the curve $\{R_k = 0\}$ contains a non singular point with rational coordinates.

Behavior near infinity

Lemma

The homogeneous part of highest degree of R_k is

$$(b-a)^{4\cdot 3^{k-2}-1}\cdot(2a+b)^{2\cdot 3^{k-2}}.$$

Corollary

The curve $\{R_k=0\}$ *intersects the line at infinity at two points:* $[1:1:0]$ *with multiplicity* $4\cdot 3^{k-2}-1$*, and* $[1:-2:0]$ *with multiplicity* $2\cdot 3^{k-2}$*.*

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- $p_{k+1} = p_k^3 + b$, $q_{k+1} = p_k^2 + p_k p_{k+1} + p_{k+1}^2$.
- $q_k = br_k = bs_k$.

Proposition (Goksel)

The polynomial $s_k \in \mathbb{Z}[b]$ is irreductible over $\mathbb{Q}$.

Proof:

- Work in $\mathbb{F}_3[b]$.
- $p_k \equiv b^{3^{k-1}} + b^{3^{k-2}} + \cdots + b^3 + b \pmod 3$.
- $p_{k+1} - p_k \equiv b^{3^k} \pmod 3$.
- $s_k \equiv b^{2\cdot 3^{k-1} - 2} \pmod 3$.

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- $s_k \equiv b^{2\cdot 3^{k-1}-2} \pmod 3$.
- Since $s_k(0) = 3$, apply the Eisenstein criterion.