Dynamics of meromorphic maps on invariant Fatou components

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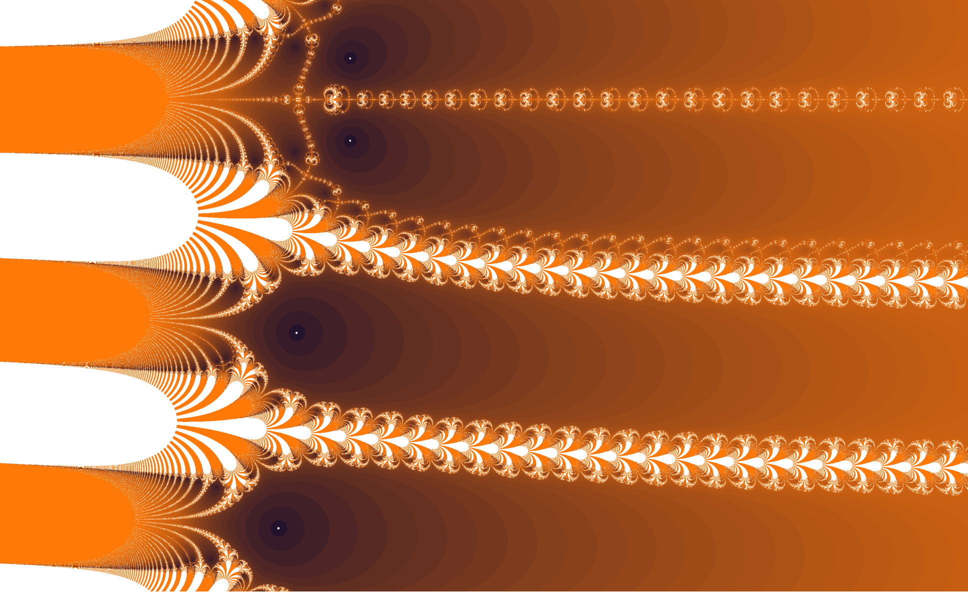
Barcelona, 27 March 2019

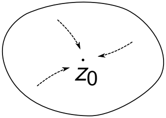


Attracting basins for the map $N(z) = z(z + e^z)/(z + 1)$,
Newton's method of $F(z) = 1 + ze^z$.

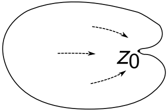
Periodic Fatou components of transcendental maps

Let $f\colon \mathbb{C} \to \overline{\mathbb{C}}$ be a **transcendental entire** or **meromorphic** map.

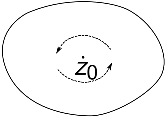
The **Fatou set** consists of points $z \in \mathbb{C}$ such that the family of iterates $\{f^n\}_{n\geq 0}$ is defined and normal in some neighbourhood of z.
Let $U \subset \mathbb{C}$ be an **periodic Fatou component**, i.e. a connected component of the Fatou sets such that $f^p(U) \subset U$ for some $p \geq 1$.
By the Classification Theorem, $f^p|_U$ has one of the following types:



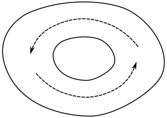
attracting basin



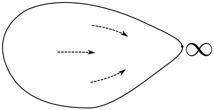
parabolic basin



Siegel disc



Herman ring



Baker domain

Lifting the map f on U

A periodic Fatou component U is a **hyperbolic domain**, i.e. a domain whose complement in $\mathbb{C}$ contains at least two points. By the Uniformization Theorem, there exists a **universal holomorphic covering** π from the open unit disc $\mathbb{D}$ onto U, and $f^p|_U$ can be lifted by π to a holomorphic map $g : \mathbb{D} \to \mathbb{D}$ with commuting diagram

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{g} & \mathbb{D} \\ \Big\downarrow \pi & & \Big\downarrow \pi \\ U & \xrightarrow{f^p} & U \end{array}$$

If U is simply connected, then π is a **Riemann map** conjugating f^p to g. For meromorphic maps U need not be simply connected!

Denjoy–Wolff Theorem

Let $g : \mathbb{D} \to \mathbb{D}$ be a non-constant holomorphic map, which is not a Möbius automorphism of $\mathbb{D}$. Then there exists the **Denjoy–Wolff point** $\zeta \in \overline{\mathbb{D}}$, *such that $g^n \to \zeta$ as $n \to \infty$ almost uniformly on $\mathbb{D}$.*

Fixed point case

Suppose f^p has a fixed point $z_0 \in U$. Then U is either an attracting basin or a Siegel disc.

Definition

A domain $W \subset U$ is **absorbing** for f^p, if $f^p(W) \subset W$ and for every compact $K \subset U$ there exists $n \geq 0$ such that $f^{pn}(K) \subset W$.

Attracting basin case

If U is an attracting basin of z_0, then there is a simply connected absorbing domain W for f^p in U (a neighbourhood of z_0), such that $\overline{f(W)} \subset W$ and $\bigcap_{n=0}^{\infty} \overline{f^n(W)} = \{z_0\}$.

Siegel disc case

If U is a Siegel disc, then there is no absorbing domain for f^p in U. Moreover, g is an elliptic Möbius automorphism of $\mathbb{D}$.

No fixed point case

Suppose f^p has no fixed point in U. Then U is a parabolic basin, Herman ring or Baker domain. Moreover, g is of **non-elliptic type**, i.e. its Denjoy–Wolff point ζ is in $\partial\mathbb{D}$. In this case we have:

Theorem (Baker–Pommerenke–Cowen ∼1980)

There exists a simply connected absorbing domain $V \subset \mathbb{D}$ for g, a domain Ω equal to the right half-plane $\mathbb{H}$ or $\mathbb{C}$, a holomorphic map $\varphi : \mathbb{D} \to \Omega$, and a Möbius transformation $T : \Omega \to \Omega$, such that $\varphi \circ g = T \circ \varphi$ on $\mathbb{D}$ and φ is univalent on V.
We have one of the three following cases:

$\Omega = \mathbb{H},$	$T(\omega) = a\omega, \ a > 1$	**hyperbolic type**
$\Omega = \mathbb{H},$	$T(\omega) = \omega \pm i$	**simply parabolic type**
$\Omega = \mathbb{C},$	$T(\omega) = \omega + 1$	**doubly parabolic type**

Parabolic basin and Herman ring case

Parabolic basin case

If U is a parabolic basin of z_0, then there exists a simply connected absorbing domain W for f^p in U (an attracting petal of z_0), such that $\overline{f(W)} \subset W \cup \{z_0\}$ and $\bigcap_{n=0}^{\infty} \overline{f^n(W)} = \{z_0\}$. Moreover, g is of doubly parabolic type.

Herman ring case

If U is a Herman ring, then there is no absorbing domain for f^p in U. Moreover, g is of hyperbolic type.

Baker domain case

Suppose U is a Baker domain. Then we have the following.

Theorem (König 1999)

Assume that every closed curve $\gamma \subset U$ is eventually contractible in U (i.e. there exists $n \geq 0$ such that $f^{pn}(\gamma)$ is contractible in U). Then there exists a simply connected absorbing domain W for f^p in U, such that $\overline{f(W)} \subset W$ and $\bigcap_{n=0}^{\infty} \overline{f^n(W)} = \emptyset$. Moreover, the above assumption is satisfied if f has at most finitely many poles.

Theorem (BFJK[1] 2014–2015)

For any Baker domain U, there exists an absorbing domain W for f^p in U, such that $\overline{f(W)} \subset W$ and $\bigcap_{n=0}^{\infty} \overline{f^n(W)} = \emptyset$. If g is of doubly parabolic type, then W can be chosen to be simply connected.

[1] Barański–Fagella–Jarque–Karpińska

Examples

Theorem (BFJK 2015)

There exist meromorphic maps of the form

$$f(z) = z + 1 + \sum_{p \in \mathcal{P}} \frac{a_p}{(z-p)^2}, \qquad a_p \in \mathbb{C} \setminus \{0\},$$

where the set of poles $\mathcal{P} \subset \mathbb{C}$ *has one of the three following forms:*

(i) $\mathcal{P} = i\mathbb{Z} = \{im : m \in \mathbb{Z}\}$,

(ii) $\mathcal{P} = \mathbb{Z}$ *or* $\mathbb{Z}_+$,

(iii) $\mathcal{P} = \mathbb{Z} + i\mathbb{Z} = \{j + im : j, m \in \mathbb{Z}\}$,

with an invariant Baker domain U *such that:*

- *in the case (i)* $f|_U$ *is of doubly parabolic type, so there exists a simply connected absorbing domain in* U *for* f*,*
- *in the case (ii)–(iii)* $f|_U$ *is not of doubly parabolic type and there is no simply connected absorbing domain in* U *for* f*.*

Accesses to boundary points

From now on, assume U is simply connected. By ∂U denote the boundary of U in $\overline{\mathbb{C}}$.
Fix $z_0 \in U$. For $v \in \partial U$ let

$$\Gamma_v = \{\gamma : [0,1) \to U : \ \gamma(0) = z_0 \text{ and } \lim_{t \to 1^-} \gamma(t) = v\}.$$

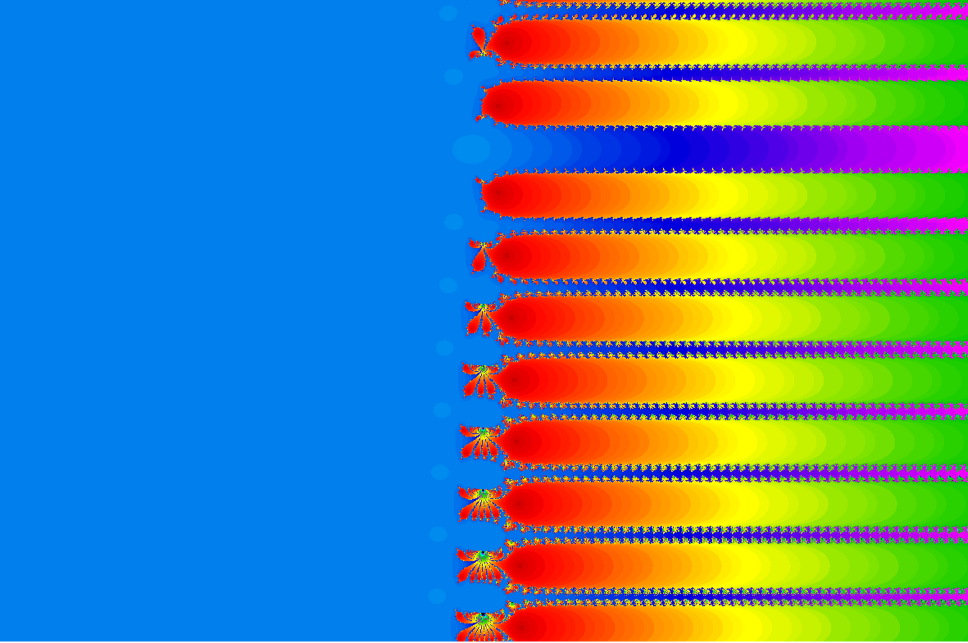
Definition

A point $v \in \partial U$ is **accessible** from U, if there exists a curve $\gamma \in \Gamma_v$. We also say that γ **lands** at v.

Let $v \in \partial U$ be accessible. An **access** to v from U is a homotopy class of curves within Γ_v.

Remark

The choice of z_0 is irrelevant for the definition of an access.



A domain with infinitely many accesses to infinity.

Correspondence between accesses and radial limits of the Riemann map

Let

$$\varphi : \mathbb{D} \to U$$

be a Riemann map from the open unit disc $\mathbb{D}$ onto U with $\varphi(0) = z_0$. We consider **radial limits** of φ at points $\zeta \in \partial\mathbb{D}$

$$\mathrm{RL}(\varphi, \zeta) = \lim_{t \to 1^-} \varphi(t\zeta)$$

(existing for a.e. $\zeta \in \partial\mathbb{D}$ by the Fatou Theorem).

Correspondence Theorem

Let $v \in \partial U$. Then there is a one-to-one correspondence between accesses to v from U and points $\zeta \in \partial\mathbb{D}$, such that $\mathrm{RL}(\varphi, \zeta)$ exists and is equal to v.

Accesses to infinity for transcendental entire maps

Theorem (Devaney–Goldberg 1987)

Let $f(z) = \lambda e^z$, $\lambda \in \mathbb{C}$, such that f has a (completely) invariant attracting basin U. Then U has uncountably many accesses to infinity, the Riemann map $\varphi : \mathbb{D} \to U$ has radial limits at all points of $\partial\mathbb{D}$, and those where the limit is equal to infinity occur at a dense set in $\partial\mathbb{D}$.

Theorem (Baker–Domínguez 1999)

For every transcendental entire map f with an invariant Fatou component U, which is not a univalent Baker domain, if U has an access to infinity, then it has infinitely many accesses to infinity. Moreover, if U is not a Baker domain, then the Riemann map $\varphi : \mathbb{D} \to U$ has radial limits equal to infinity on a dense set of $\partial\mathbb{D}$.

Accesses to infinity for entire maps in class $\mathcal{B}$

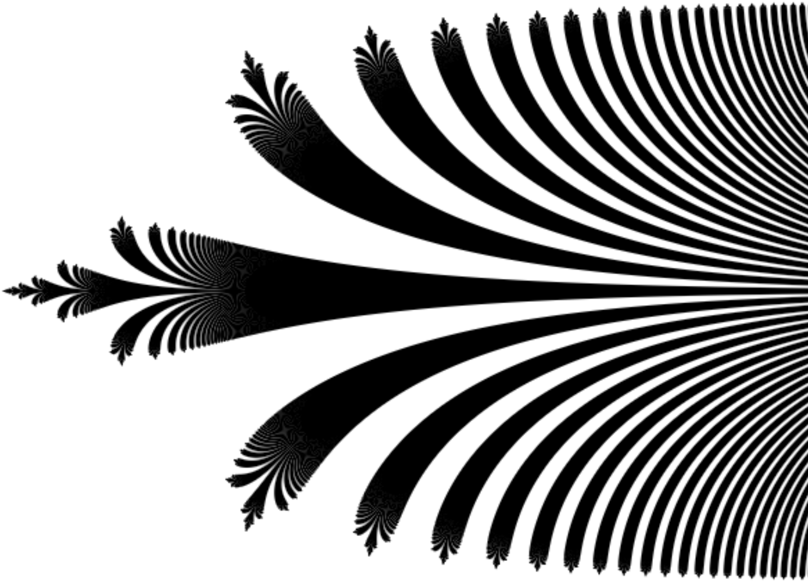
The **Eremenko–Lyubich class** is defined as

$$\mathcal{B} = \{f : \text{the set of singularities of } f^{-1} \text{ is bounded}\}.$$

Theorem (KB[2] 2007)

For every transcendental entire map f from class $\mathcal{B}$ of disjoint type (i.e. hyperbolic with a unique (attracting) Fatou component U), the domain U has uncountably many accesses to infinity and the Riemann map $\varphi : \mathbb{D} \to U$ has unrestricted limits equal to infinity on a dense set of $\partial\mathbb{D}$.
If, additionally, f has finite order, then φ has radial limits at all points of $\partial\mathbb{D}$.

[2] Karpińska–Barański or Krzysztof–Bogusia



Accesses to infinity from an attracting basin of an exponential map.

Invariant accesses

For simplicity, assume U is invariant. Fix a curve η connecting z_0 to $f(z_0)$ in U. Let $v \in \partial U$.

Definition

Let $\mathcal{A}$ be an access to v from U.
$\mathcal{A}$ is **invariant**, if $f(\gamma) \cup \eta \in \mathcal{A}$ for some $\gamma \in \mathcal{A}$.
$\mathcal{A}$ is **strongly invariant**, if $f(\gamma) \cup \eta \in \mathcal{A}$ for every $\gamma \in \mathcal{A}$.

Remark

Since U is simply connected, the choice of the curve η is irrelevant.

Remark

Periodic accesses can be defined analogously. These are related to landing periodic rays in simply connected invariant Fatou components, e.g. basins of infinity for polynomials.

Invariant vs. strongly invariant accesses

Example (Invariant access which is not strongly invariant)

Let $f\colon \mathbb{C} \to \overline{\mathbb{C}}$, $f(z) = z + \tan z$ and $U = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$. Then U is an invariant Baker domain of f and the access to infinity defined by the curve $i\mathbb{R}_+$ is invariant, but not strongly invariant.

Example (Strongly invariant access)

Let $f\colon \mathbb{C} \to \overline{\mathbb{C}}$, $f(z) = z - \tan z$, Newton's method of $F(z) = \sin z$. Then f has infinitely many basins of attraction U_k, $k \in \mathbb{Z}$, of superattracting fixed points $k\pi$, and every U_k has two strongly invariant accesses to infinity.

Characterization of strongly invariant accesses

Let $\mathcal{A}$ be an invariant access to v from U. Then $\mathcal{A}$ is strongly invariant if and only if for every $\gamma \in \mathcal{A}$ the curve $f \circ \gamma$ lands at some point in ∂U.

Dynamical invariant access

Example

Every simply connected invariant Baker domain U has a **dynamical invariant access** to infinity, defined by the curve $\gamma = \bigcup_{n\geq 0} f^n(\eta)$, where η joins a point $z_0 \in U$ to $f(z_0)$ within U. Similarly, every simply connected invariant parabolic basin U has a dynamical invariant access to a parabolic fixed point $v \in \partial U$.

Newton maps

Definition

Let $F : \mathbb{C} \to \mathbb{C}$ be an entire map. A **Newton map** is the meromorphic map

$$N(z) = z - \frac{F(z)}{F'(z)},$$

which is Newton's method of finding zeroes of F.

Properties of Newton maps

The fixed points of N in $\mathbb{C}$ are precisely the zeroes of F, and all of them are attracting. For rational Newton maps (F polynomial or $F = pe^q$, p, q polynomials), infinity is a weakly repelling fixed point, otherwise it is an essential singularity. This implies that Newton maps have no Siegel discs and no fixed points in the Julia set.

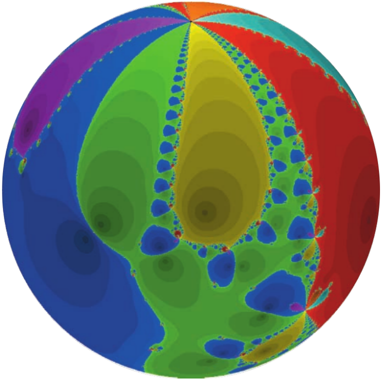
Theorem (Shishikura 2009, BFJK 2014)

All Fatou components of a Newton map are simply connected.

Accesses to infinity for rational Newton maps

Theorem (Hubbard–Schleicher–Sutherland 2001)

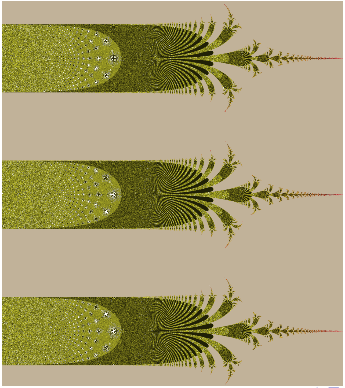
If F is a polynomial and U is an invariant basin of attraction to a zero of F for the Newton's method N applied to F, then the number of accesses to infinity from U is finite and equal to the number of critical points of N in U, counted with multiplicity.



Accesses to infinity for transcendental Newton maps

Example (Baker–Domínguez 1999)

Let $N(z) = z + e^{-z}$, Newton's method applied to $F(z) = e^{-e^z}$. Then N has infinitely many invariant Baker domains U_k, $k \in \mathbb{Z}$, such that $U_k = U_0 + 2k\pi i$, $\deg N|_{U_k} = 2$ and U_k has infinitely many accesses to infinity.



Accesses to infinity for transcendental Newton maps

Theorem (BFJK 2017)

(a) *If* $\deg N|_U = d < \infty$*, then there are no invariant accesses from* U *to points* $v \in \partial U \cap \mathbb{C}$ *and exactly* D *invariant accesses from* U *to infinity, where* D *is the number of fixed points of* g *in* $\partial\mathbb{D}$*. Moreover,* $D \geq 1$ *and* $d - 1 \leq D \leq d + 1$*.*

(b) *If* $\deg N|_U = \infty$ *and* $N|_U$ *is* **singularly nice***, then there are infinitely many invariant accesses to infinity from* U*.*

Remark

$N|_U$ is singularly nice e.g. if at least one **singularity** of the associated **inner function**

$$g : \mathbb{D} \to \mathbb{D}, \qquad g = \varphi^{-1} \circ N|_U \circ \varphi$$

is isolated. A point $\zeta^* \in \partial\mathbb{D}$ is a singularity of g, if g does not extend analytically to any neighbourhood of ζ^*.

Accesses for Newton maps with a completely invariant Fatou component

Theorem (BFJK 2017)

If N has a completely invariant Fatou component $V \neq U$, then

$$\deg N|_U \in \{1, 2, \infty\}$$

and for every $v \in \partial U$ there is at most one access to v from U. Moreover,

(a) *if $\deg N|_U \in \{1, 2\}$, then U has a unique access $\mathcal{A}$ to ∞ and $\mathcal{A}$ is invariant,*

(b) *if $\deg N|_U = 1$, then ∂U does not contain a pole of N accessible from U,*

(c) *if $\deg N|_U = 2$, then ∂U contains exactly one accessible pole of N.*

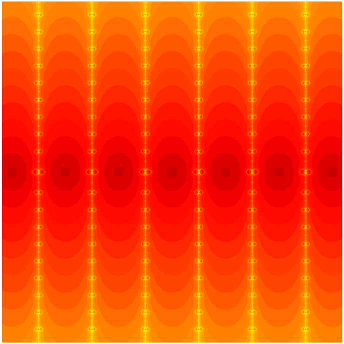
Example 1

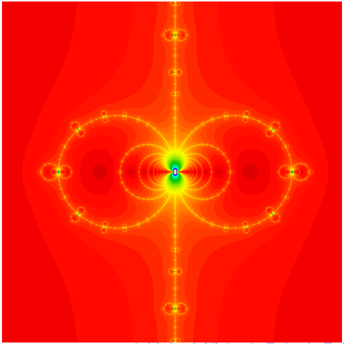
Let $N(z) = z - \tan z$, Newton's method of $F(z) = \sin z$. Then:

(a) N has infinitely many immediate basins of attraction U_k, $k \in \mathbb{Z}$, such that $U_k = U_0 + k\pi$ and $\deg N|_{U_k} = 3$.

(b) Each basin U_k has exactly two accesses to infinity, and they are strongly invariant.

(c) ∂U_k contains exactly two accessible poles of N.





Example 2

Let $N(z) = z + i + \tan z$, Newton's method of $F(z) = \exp\left(-\int_0^z \frac{du}{i+\tan u}\right)$. Then:

(a) N has a completely invariant Baker domain U, in particular $\deg N|_U = \infty$.

(b) U has infinitely many strongly invariant accesses to infinity and the dynamical access to infinity from U is invariant but not strongly invariant.

(c) ∂U contains infinitely many accessible poles of N.

(d) The inner function associated to $N|_U$ has a unique singularity in $\partial\mathbb{D}$, so $N|_U$ is singularly nice.

(e) N has infinitely many invariant Baker domains U_k, $k \in \mathbb{Z}$, such that $U_k = U_0 + k\pi$ and $\deg N|_{U_k} = 2$.

(f) U_k has exactly one access to infinity, and it is strongly invariant.

(g) ∂U_k contains exactly one accessible pole of N.

Example 3

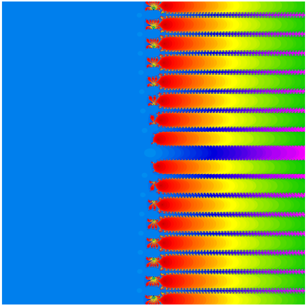
Let $N(z) = \frac{e^z(z-1)}{e^z+1}$, Newton's method of $F(z) = z + e^z$. Then:

(a) N has a completely invariant immediate superattracting basin U_0, in particular $\deg N|_{U_0} = \infty$.

(b) U_0 has infinitely many invariant accesses to infinity.

(c) The inner function associated to $N|_{U_0}$ has a unique singularity, so $N|_{U_0}$ is singularly nice.

(d) N has infinitely many superattracting basins U_k, $k \in \mathbb{Z} \setminus \{0\}$, such that U_k has at most one access to infinity.



Thank you for attention!

Theorem (BFJK 2017)

Suppose infinity is accessible from U. Set $d = \deg f|_U$. Then:

(a) *If $1 < d < \infty$ and $\partial U \cap \mathbb{C}$ contains no poles of f accessible from U, then U has infinitely many accesses to infinity, from which at most $d + 1$ are invariant.*

(b) *If $d = \infty$ and ∂U contains only finitely many poles of f accessible from U, then U has infinitely many accesses to infinity.*

Inner functions associated to Fatou components

Let

$$g : \mathbb{D} \to \mathbb{D}, \qquad g = \varphi^{-1} \circ f \circ \varphi.$$

Then g is the **inner function** associated to $f|_U$, with radial limits belonging to $\partial\mathbb{D}$ at almost every point of $\partial\mathbb{D}$.

Definition

A point $\zeta^* \in \partial\mathbb{D}$ is a **singularity** of g, if g does not extend analytically to any neighbourhood of ζ^*.

Remark

If $\deg g < \infty$, then g is a finite Blaschke product which extends to $\overline{\mathbb{C}}$.
If $\deg g = \infty$, then g has at least one singularity in $\partial\mathbb{D}$.

Inner functions and boundary fixed points

Definition

A point $\zeta \in \partial\mathbb{D}$ is a (radial) **boundary fixed point** of the inner function g, if $\mathrm{RL}(g, \zeta) = \zeta$.

Fact

By the Julia–Wolff Lemma, at every boundary fixed point ζ the map g has an **angular derivative** $\lim_{t \to 1^-} \frac{g(t\zeta) - \zeta}{t\zeta - \zeta}$, which is either a positive real number or infinity.

Definition

A boundary fixed point $\zeta \in \partial\mathbb{D}$ of g is called **regular**, if the angular derivative of g at ζ is finite.

Accesses and boundary fixed points

Definition

A fixed point z of f is **weakly repelling**, if $|f'(z)| > 1$ or $f'(z) = 1$.

Theorem (BFJK 2017)

(a) *If $\mathcal{A}$ is an invariant access from U to a point $v \in \partial U$, then v is either infinity or a fixed point of f and $\mathcal{A}$ corresponds to a boundary fixed point $\zeta \in \partial\mathbb{D}$ of the inner function g.*

(b) *If $\zeta \in \partial\mathbb{D}$ is a regular boundary fixed point of g, then $\mathrm{RL}(\varphi, \zeta)$ exists and is equal to v, where v is either infinity or a weakly repelling fixed point of f in ∂U. Moreover, ζ corresponds to an invariant access to v from U.*

Singularly nice maps

Definition

The map $f|_U$ is **singularly nice** if there exists a singularity $\zeta^* \in \partial\mathbb{D}$ of the inner function g associated to $f|_U$, such that the angular derivative of g is finite at every point ζ in some punctured neighbourhood of ζ^* in $\partial\mathbb{D}$.

Remark

If g has an isolated singularity in $\partial\mathbb{D}$, then $f|_U$ is singularly nice.

Proposition

If $\deg f|_U = \infty$ and there exists a non-empty open set $W \subset U$, such that for every $z \in W$ the set $f^{-1}(z) \cap U$ is contained in the union of a finite number of curves in U landing at some points of ∂U, then $f|_U$ is singularly nice.

Let

$$\begin{aligned} \mathrm{IA}(U) &= \{\text{invariant accesses from } U \text{ to its boundary points}\} \\ \mathrm{IA}(\infty, U) &= \{\text{invariant accesses from } U \text{ to infinity}\} \\ \mathrm{IA}(\mathrm{wrfp}, U) &= \{\text{invariant accesses from } U \text{ to weakly repelling fixed} \\ &\qquad \text{points of } f \text{ in } \partial U\}. \end{aligned}$$

Theorem (BFJK 2017)

(a) *If* $\deg f|_U = d < \infty$, *then* $\mathrm{IA}(U) = \mathrm{IA}(\infty, U) \cup \mathrm{IA}(wrfp, U)$ *and* $\mathrm{IA}(U)$ *has exactly* D *elements, where* D *is the number of fixed points of* g *in* $\partial\mathbb{D}$. *Moreover,* $D \geq 1$ (*unless* U *is an invariant Siegel disc*) *and* $d - 1 \leq D \leq d + 1$.

(b) *If* $\deg f|_U = \infty$ *and* $f|_U$ *is singularly nice, then* $\mathrm{IA}(\infty, U) \cup \mathrm{IA}(wrfp, U)$ *is infinite.*

(c) *If* U *is bounded, then* f *has a weakly repelling fixed point in* ∂U *accessible from* U, *or* U *is an invariant Siegel disc.*

Three lemmas to prove theorem on accesses

Lemma 1

Let $U \subset \mathbb{C}$ be a simply connected domain. Suppose that $\gamma_0, \gamma_1 : [0,1) \to U$ are curves such that $\gamma_0(0) = \gamma_1(0) = z_0$ and γ_0 lands at $v \in \partial U$. If there exists $c > 0$ such that $\varrho_U(\gamma_0(t), \gamma_1(t)) < c$ for every $t \in [0,1)$, then γ_1 lands at v and γ_0, γ_1 are in the same access to v from U.

Lemma 2

Let $g : \mathbb{D} \to \mathbb{D}$ be a holomorphic map and $\zeta \in \partial\mathbb{D}$ a regular boundary fixed point of g. Then there exists $c > 0$ such that $\varrho_{\mathbb{D}}(g(t\zeta), t\zeta) < c$ for every $t \in [0,1)$.

Lemma 3 (Modified Snail Lemma)

Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a meromorphic map and U a simply connected invariant Fatou component of f. Suppose that a curve $\gamma : [0,1) \to U$ lands at a fixed point $v \in \partial U$ of f and there exists $c > 0$ such that $\varrho_U(f(\gamma(t)), \gamma(t)) < c$ for every $t \in [0,1)$. Then v is a weakly repelling fixed point of f.