Fingers in the parameter space of the complex standard family

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– joint work with Mitsuhiro Shishikura –

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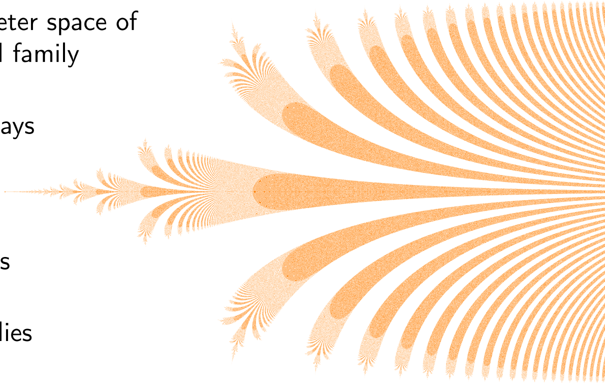
Sketch of the talk

1. Introduction to the Arnol'd standard family
2. Fingers in the parameter space of the complex standard family
3. Invariant dynamical rays and parameter rays
4. Parabolic implosion and number of fingers
5. Fingers in other families



The Arnol'd standard family

The Arnol'd **standard family** of circle maps is given by, for $\alpha, \beta \in \mathbb{R}$,

$$F_{\alpha,\beta}(\theta) := \theta + \alpha + \beta \sin\theta \pmod{2\pi}, \qquad \text{for } \theta \in [0, 2\pi),$$

and are transcendental perturbations of the rigid rotation of angle α

$$F_{\alpha,0}(\theta) = \theta + \alpha \pmod{2\pi}, \qquad \text{for } \theta \in [0, 2\pi).$$

Arn61 V. I. Arnol'd, *Small denominators I. Mapping the circle onto itself*. Izv. Akad. Nauk SSSR Ser. Mat. **25** 1961 21–86.

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For $|\beta| < 1$, the map $F_{\alpha,\beta}$ is an orientation preserving homeomorphism of the circle.

Let $\theta \in \mathbb{R}$, the **rotation number** of $F_{\alpha,\beta}$ is given by

$$\omega(F_{\alpha,\beta}) := \lim_{n\to\infty} \frac{F^n_{\alpha,\beta}(\theta) - \theta}{n} \in [0, 2\pi).$$

The rigid rotation of angle α has rotation number equal to α.

Arn61 V. I. Arnol'd, *Small denominators I. Mapping the circle onto itself*. Izv. Akad. Nauk SSSR Ser. Mat. **25** 1961 21–86.

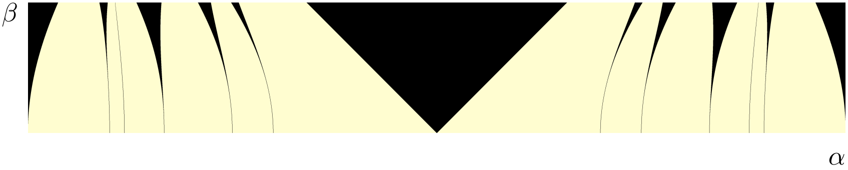
Arnol'd tongues

To study the dependence of the rotation number on the parameters (α, β), for $\rho \in [0, 2\pi)$ Arnol'd considered the sets of parameters

$$T_\rho := \{(\alpha, \beta) \in \mathbb{R}^2 \ : \ \omega(F_{\alpha,\beta}) = \rho\}$$

which are known as the **Arnol'd tongues** and satisfy that:

- if $\rho \in \mathbb{Q}$, then T_ρ has non-empty interior,
- if $\rho \in \mathbb{R} \setminus \mathbb{Q}$, then T_ρ is a curve.



The boundaries of the tongues are analytic curves and the tongue T_0 of rotation number $\rho = 0$ has boundaries given by $\alpha = \pm\beta$.

Arn61 V. I. Arnol'd, *Small denominators I. Mapping the circle onto itself*. Izv. Akad. Nauk SSSR Ser. Mat. **25** 1961 21–86.

The complex Arnol'd standard family

The Arnol'd standard family can be extended to a family of **transcendental self-maps of the punctured plane** $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$

$$f_{\alpha,\beta}(z) := ze^{i\alpha} e^{\beta(z-1/z)/2},$$

which has as lifts the family of transcendental entire functions

$$F_{\alpha,\beta}(z) := z + \alpha + \beta \sin z,$$

that is

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{F_{\alpha,\beta}} & \mathbb{C} \\ {\scriptstyle e^{iz}} \downarrow & & \downarrow {\scriptstyle e^{iz}} \\ \mathbb{C}^* & \xrightarrow{f_{\alpha,\beta}} & \mathbb{C}^* \end{array}$$

This is known as the **complex standard family** and the iteration of these functions was studied for the first time by Fagella in her PhD thesis.

Fag99 N. Fagella, *Dynamics of the complex standard family*. J. Math. Anal. Appl. **229** (1999), no. 1, 1–31.

The α-parameter space

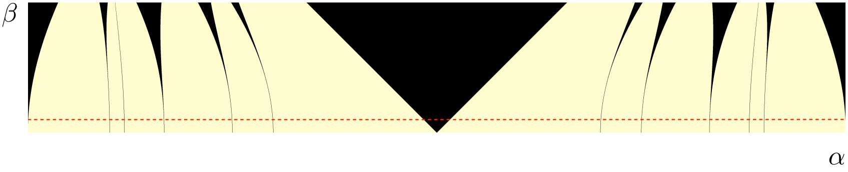
We fix the parameter $0 < \beta < 1$ and **study the bifurcation with respect to the parameter** $\alpha \in \mathbb{C}$. Note that this is **not a natural parameter space**.

We can restrict to the vertical band $B_0 := \{z \in \mathbb{C} \ : \ -\pi \leqslant \operatorname{Re} z < \pi\}$ as

$$F_{\alpha,\beta}(z + 2\pi) = F_{\alpha,\beta}(z) + 2\pi,$$

and thus the α-parameter space is 2π-**periodic**.

Observe that the **real axis** of the α-parameter space corresponds to the line at height β in the real parameter space where the Arnol'd tongues lie.



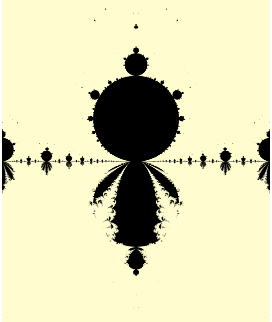
The critical orbits

For $0 < \beta < 1$, the function $F_{\alpha,\beta}$ has two critical points

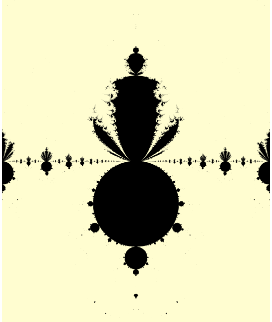
$$c^0_{\pm} = -\pi \pm i\,\mathrm{arccosh}(1/\beta)$$

in the vertical band B_0 that are complex conjugates and their orbits satisfy

$$F^n_{\alpha,\beta}(c^0_+) = \overline{F^n_{\overline{\alpha},\beta}(c^0_-)}, \quad \text{for all } n \in \mathbb{N}_0.$$



Iteration of c^0_+
for $\alpha \in \mathbb{C}$ and $\beta = 0.1$

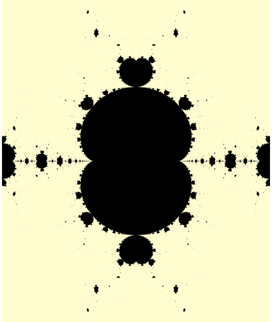


Iteration of c^0_-
for $\alpha \in \mathbb{C}$ and $\beta = 0.1$

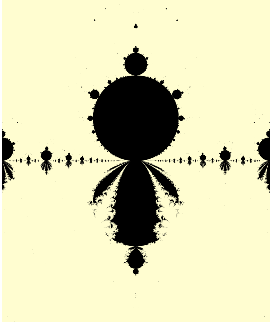
Finger-like structures

When $\beta = 1$, the α-parameter space of the complex standard family is symmetric with respect to the real axis.

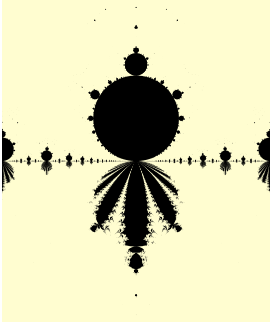
As we let $\beta \to 0$, we can observe an increasing number of finger-like structures appearing in the lower half plane, which seem to be contained in the reflection of the set in the upper half plane.



$\beta = 1$



$\beta = 0.1$



$\beta = 0.01$

Limiting dynamics as $\beta \to 0$

If we set $\beta = 0$, then $F_{\alpha,0}(z) = z + \alpha$, the dynamics of which is trivial. However, Fagella showed that the dynamics of $F_{\alpha,\beta}$ do not become trivial as $\beta \to 0$. She proved that we can rescale $F_{\alpha,\beta}$ by setting

$$\tilde{z} = z + i \log(2/\beta)$$

and, in this variable, the function $F_{\alpha,\beta}$ becomes

$$\tilde{F}_{\alpha,\beta}(\tilde{z}) = \tilde{z} + \alpha - ie^{i\tilde{z}} + i\frac{\beta^2}{4}e^{-i\tilde{z}}.$$

When we make $\beta \to 0$, we obtain the one parameter family

$$\tilde{F}_{\alpha,\beta}(\tilde{z}) \to \tilde{z} + \alpha - ie^{i\tilde{z}} =: G_\alpha(\tilde{z})$$

which are lifts of the family of transcendental self-maps of $\mathbb{C}^*$

$$g_\lambda(z) = \lambda z e^z,$$

where $\lambda = e^{i\alpha}$.

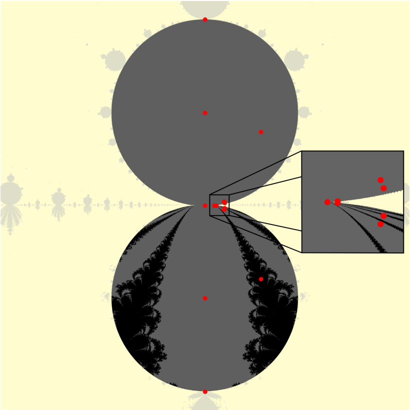
Fag95 N. Fagella, *Limiting dynamics for the complex standard family*. Internat. J. Bifur. Chaos Appl. Sci. Engrg. **5** (1995), 3, 673–699.

The region $\mathcal{A}_\beta$

We fix $0 < \beta < 1$ and focus our study in the set of parameters

$$\mathcal{A}_\beta := \{\alpha \in \mathbb{C} \ : \ \text{the function } F_{\alpha,\beta} \text{ has an attracting fixed point } \xi\}$$

and for such α, one critical point of $F_{\alpha,\beta}$ lies in the immediate attracting basin of ξ while the other one is free.



Definition of the fingers

For $0 < \beta < 1$ and $\alpha \in \mathcal{A}_\beta$, the function $F_{\alpha,\beta}$ has an attracting and a repelling fixed point in each vertical band $B_n = B_0 + 2n\pi$. Let U_n be the immediate basin of attraction of the attracting fixed point that lies in B_n.

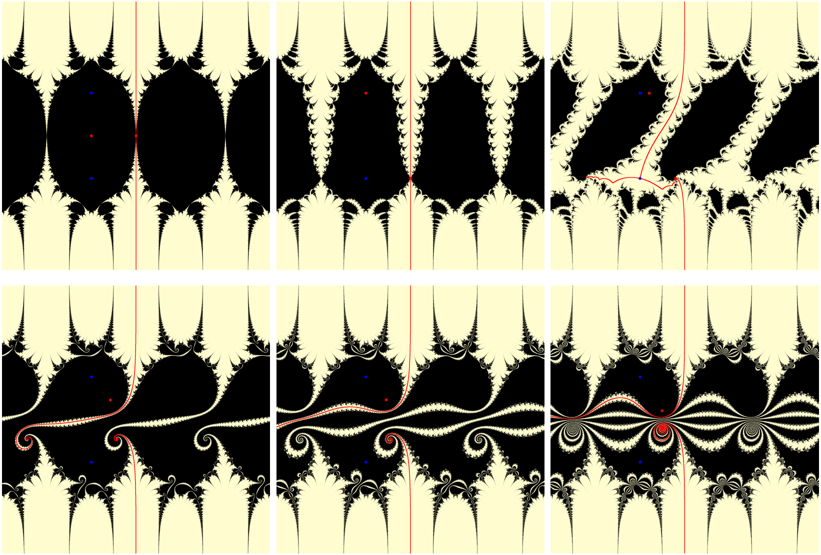
For $n \in \mathbb{Z}$, we define the nth **finger** in $\mathcal{A}_\beta$ as the subset

$$\mathcal{T}_\beta^n := \{\alpha \in \mathcal{A}_\beta \ : \ c_-^0 \in U_n\}.$$

By definition, the fingers $\mathcal{T}_\beta^n$ are open sets.

Question: Are the sets $T_\beta^n \neq \emptyset$ for all $n \in \mathbb{Z}$?

Dynamics in the fingers

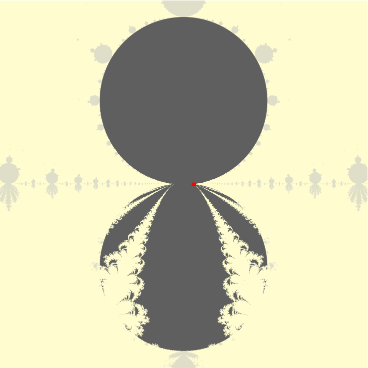


The parabolic map f_0

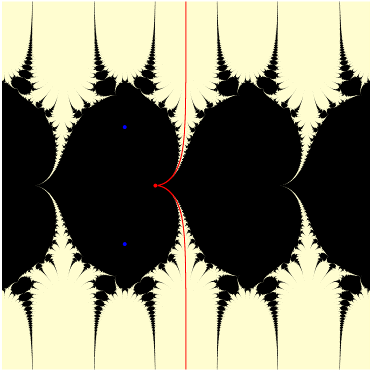
When $\alpha = \beta$, the map

$$f_0(z) := z + \alpha + \beta \sin z = z + \beta(1 + \sin z)$$

has a **parabolic fixed point** at $z_0 = -\frac{\pi}{2}$ with $f_0'(z_0) = 1$.



Parameter space $\mathcal{A}_\beta$
with $\beta = 0.1$



Dynamical plane of f_0
with $\beta = 0.1$

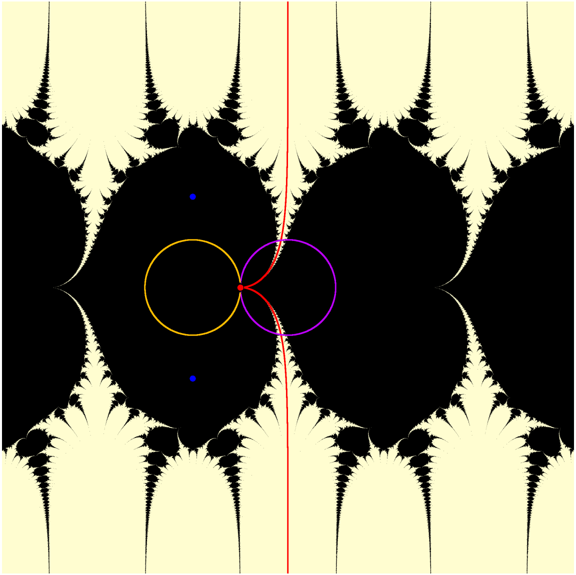
Leau-Fatou flower theorem

Since $f_0'(z_0) = e^{2\pi ip/q}$ with $p = 0,\ q = 1$, by the **Leau-Fatou flower theorem** there exist

an **attracting petal** S_- such that $f_0(S_-) \subseteq S_-$

and

a **repelling petal** S_+ such that $f_0(S_+) \supseteq S_+$.



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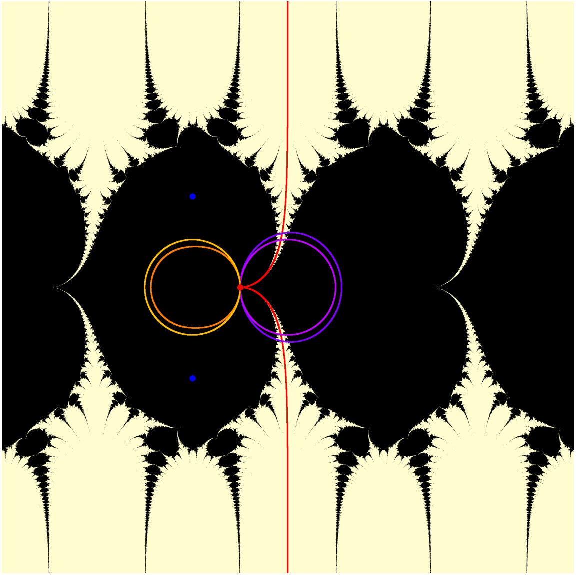

Fatou coordinates

There exist two univalent maps

$$\Phi_{\text{attr}} : V_- \to \mathbb{C} \quad \text{and} \quad \Phi_{\text{rep}} : V_+ \to \mathbb{C}$$

such that

$$\Phi_{\text{attr}}(f_0(z)) = \Phi_{\text{attr}}(z) + 1 \quad \text{and} \quad \Phi_{\text{rep}}(f_0(z)) = \Phi_{\text{rep}}(z) + 1$$

whenever $z \in V_\pm$ and $f_0(z) \in V_\pm$. We can quotient by the dynamics and obtain maps

$$\tilde{\Phi}_{\text{attr}} : V_- \to \mathbb{C}/\mathbb{Z} \quad \text{and} \quad \tilde{\Phi}_{\text{rep}} : V_+ \to \mathbb{C}/\mathbb{Z}.$$

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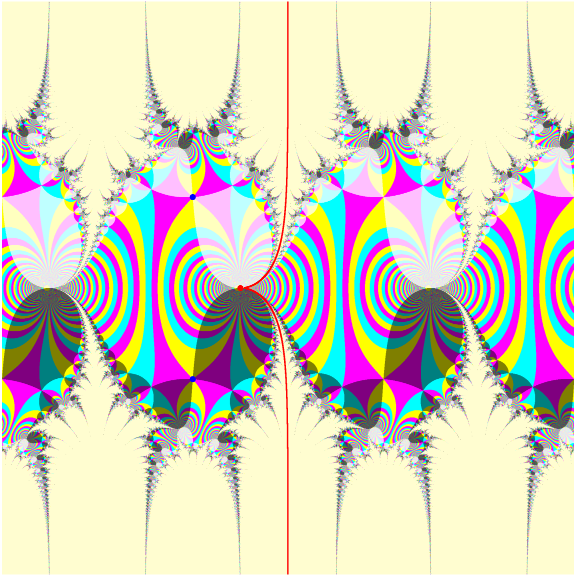
There exists a **horn map** from the repelling cylinder to the attracting cylinder which is a branched covering

$$E_{f_0} : \text{Dom}(E_{f_0}) \setminus f_0^{-1}(\{v_-, v_+\}) \to \mathbb{C}/\mathbb{Z} \setminus \{v_-, v_+\}$$

and $\text{Dom}(E_{f_0})$ has **3 components** that contain the real axis and the two ends of the cylinder.

The parabolic checkerboard



After perturbation

Let us now consider the maps

$$f_\varepsilon(z) = f_0(z) + \varepsilon = z + \alpha + \beta \sin z,$$

that is, $\varepsilon = \alpha - \beta$.

After perturbation, Fatou coordinates can still be defined: there exist maps

$$\Phi^\varepsilon_{\text{attr}} : V^\varepsilon_- \to \mathbb{C} \quad \text{and} \quad \Phi^\varepsilon_{\text{rep}} : V^\varepsilon_+ \to \mathbb{C}$$

such that

$$\Phi^\varepsilon_{\text{attr}}(f_0(z)) = \Phi^\varepsilon_{\text{attr}}(z) + 1 \quad \text{and} \quad \Phi^\varepsilon_{\text{rep}}(f_0(z)) = \Phi^\varepsilon_{\text{rep}}(z) + 1$$

whenever $z \in V^\varepsilon_\pm$ and $f_0(z) \in V^\varepsilon_\pm$. As before, there exists a horn map E_{f_ϵ} from the repelling cylinder to the attracting cylinder.

Now there exists a map χ_ε from the attracting cylinder to the repelling cylinder

$$\chi_\varepsilon(z) = z - \frac{\pi}{\sqrt{\varepsilon}} + o(1)$$

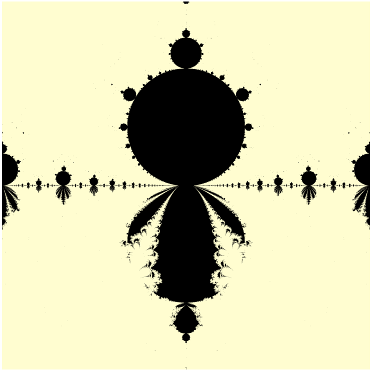
which allows us to identify both cylinders.

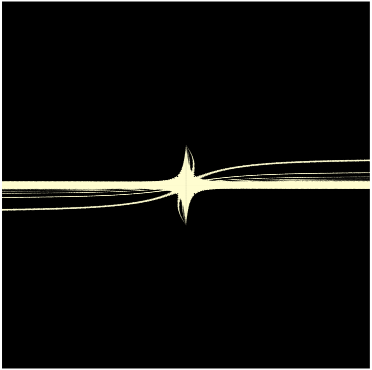
Elephants

Consider the new parameter γ given by

$$\alpha = \beta + \frac{\pi^2}{\gamma^2}$$

so that $\chi_\varepsilon(z) = z + \gamma + o(1)$.





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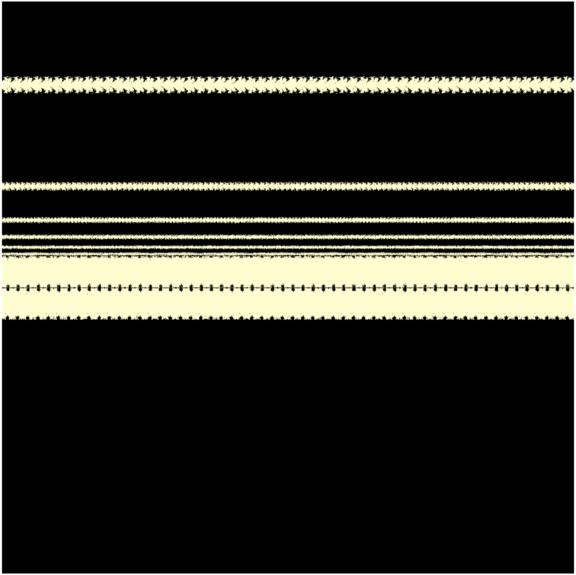

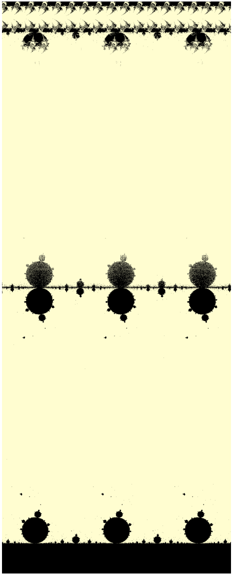

Estimating the number of fingers

Consider the constants

$$h = \sqrt{\frac{\beta}{2}} \quad \text{and} \quad \eta = \operatorname{Im} v_+ - \operatorname{Im} v_-,$$

then the number of fingers is given by the number of $k \in \mathbb{N}$ such that

$$\operatorname{Im} \gamma = \eta / k > h.$$

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$$\eta/6 \simeq 0.2372$$
$$\eta/7 \simeq 0.2033 < h$$

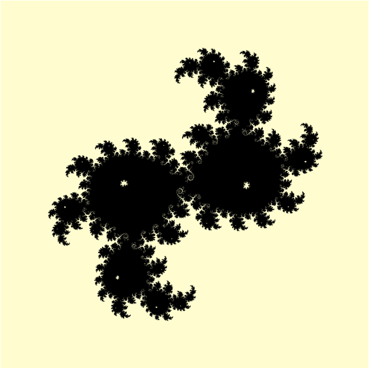
therefore in this case we have **6 fingers** to each side of the central finger.

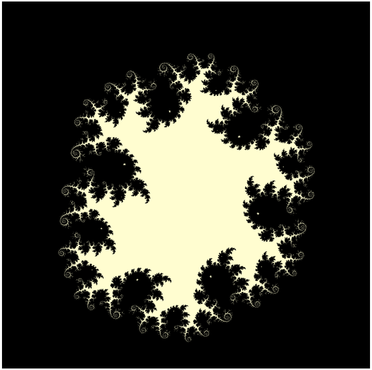
A family of Blaschke products

For $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{R}$, consider the family of rational functions

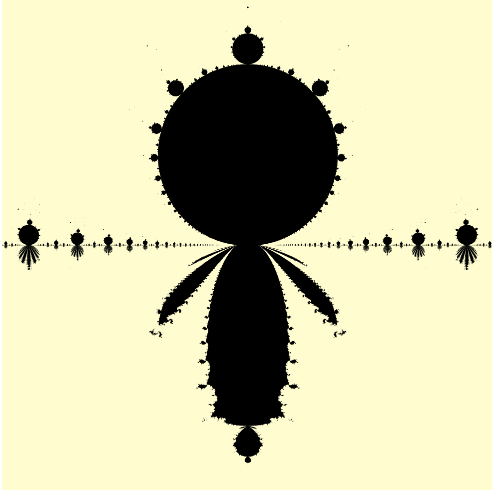
$$B_{\alpha,\beta}(z) := e^{\alpha i} z^2 \frac{1 + \beta z}{z + \beta}$$

such that $B_{\alpha,\beta}(0) = 0$, $B_{\alpha,\beta}(-\beta) = \infty$ and, for $\alpha \in \mathbb{R}$, $B_{\alpha,\beta}$ maps the unit circle to itself.





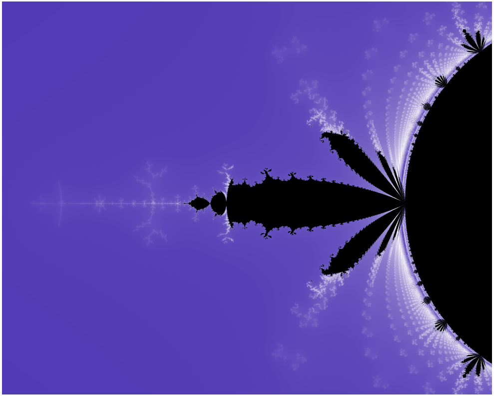
Fingers for Blaschke products



The α-parameter space of the family $B_{\alpha,\beta}$ for $\beta = 0.01$.

Fingers for cubic polynomials and Hénon maps

Finger-like structures were observed for the first time by Hubbard in the study of Hénon maps in $\mathbb{C}^2$. Motivated by this, Radu and Tanase studied the family of cubic maps and also observed the existence of similar finger-like structures.



Picture of the fingers for Henon maps by Radu and Tanase.

Thank you
for your
attention!