Cauchy quotient means and their properties

Martin Himmel

Department of Mathematics, Computer Science and Econometrics University of Zielona Góra

Topics in Complex Dynamics, October 2-6, 2017, Barcelona

Joint work with Janusz Matkowski



A B A A B A

Introduction

- 2 Functional equations and means
- 3 Means in terms of beta-type functions
- Properties of beta-type functions and its mean
- **5** A characterization of \mathfrak{B}_k in the class of premeans of beta-type
- 6 Affine functions with respect to \mathfrak{B}_k

Functional equations: the Cauchy equation

$$f(x+y) = f(x) + f(y), \qquad x, y \in \mathbb{R},$$

(Additive Cauchy equation)

▶ ★ 문 ▶ ★ 문 ▶ _ 문[님

Functional equations: the Cauchy equation

 $f(x+y) = f(x) + f(y), \qquad x, y \in \mathbb{R},$ (Additive Cauchy equation) $f(x+y) = f(x)f(y), \qquad x, y \in \mathbb{R},$ (Exponential Cauchy equation) $f(xy) = f(x)f(y), \qquad x, y > 0,$ (Multiplicative Cauchy equation) $f(xy) = f(x) + f(y), \qquad x, y > 0,$ (Logarithmic Cauchy equation)

< □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □

A functional equation related to the Gamma function

$$f(x+1) = xf(x), \qquad x > 0,$$

315

▲圖 ▶ ▲ 臣 ▶ ▲ 臣 ▶

A functional equation related to the Gamma function

$$f(x+1)=xf(x), \qquad x>0,$$

Sine addition formula

$$f(x+y) = f(x)g(y) + f(y)g(x), \qquad x, y \in \mathbb{R},$$

Jensen equation

$$f(\frac{x+y}{2}) = \frac{f(x)+f(y)}{2}.$$

비로 서로에 서로에 비해

What everybody knows about means..

The arithmetic mean $A: \mathbb{R}^2 \to \mathbb{R}$

$$A(x,y) = rac{x+y}{2}, \qquad x,y \in \mathbb{R},$$

(本部) (本語) (本語) (美) (本)

What everybody knows about means..

The arithmetic mean $A:\mathbb{R}^2
ightarrow\mathbb{R}$

$$A(x,y)=rac{x+y}{2},\qquad x,y\in\mathbb{R},$$

The geometric mean $G:(0,\infty)^2
ightarrow (0,\infty)$

$$G(x,y) = \sqrt{xy}, \qquad x, y > 0,$$

The harmonic mean

$$H(x,y)=\frac{2xy}{x+y},$$

AGH inequality

$$A(x,y) \ge G(x,y) \ge H(x,y), \qquad x,y > 0.$$

御 🛛 🗸 문 🗸 문 🗸 문법

Definition 1.

Let $I \subseteq \mathbb{R}$ be a non-empty interval, $k \in \mathbb{N}$, $k \ge 2$, and $M : I^k \to \mathbb{R}$. The function M is called a mean in the interval I, if

$$\min(x_1,\ldots,x_k) \leq M(x_1,\ldots,x_k) \leq \max(x_1,\ldots,x_k)$$

holds true for all $x_1, \ldots, x_k \in I$.

· · · · · · · · ·

Beta-type functions

Motivated by the relationship between the Euler Gamma function $\Gamma: (0,\infty) \to (0,\infty)$ and the Beta function $B: (0,\infty)^2 \to (0,\infty)$

$$\mathcal{B}(x,y) = rac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \qquad x,y > 0,$$

we introduce a new class of functions, called beta-type functions.

Beta-type functions

Motivated by the relationship between the Euler Gamma function $\Gamma: (0,\infty) \to (0,\infty)$ and the the Beta function $B: (0,\infty)^2 \to (0,\infty)$

$$\mathcal{B}(x,y) = rac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \qquad x,y > 0.$$

we introduce a new class of functions, called beta-type functions.

Definition [Himmel, Matkowski 2015]

Let $a \ge 0$ be fixed. For $f : (a, \infty) \to (0, \infty)$, the two variable function $B_f : (a, \infty)^2 \to (0, \infty)$ defined by

$$B_f(x,y) = \frac{f(x)f(y)}{f(x+y)}, \qquad x,y > a,$$

is called the *beta-type function*, and *f* is called its *generator*.

With this definition we have: $B_{\Gamma} = \mathcal{B}$.

Means and beta-type functions

We are interested in answering when the beta-type function is a bivariable mean. The answer is given in the following

Theorem 2.

Let $f : (0, \infty) \to (0, \infty)$ be an arbitrary function. The following conditions are equivalent:

(i) the beta-type function $B_f: (0,\infty)^2 \to (0,\infty)$ is a bivariable mean, i.e.

$$\min\left(x,y
ight)\leq B_{f}\left(x,y
ight)\leq \max\left(x,y
ight),\qquad x,y>0;$$

(ii) there is an additive function $\alpha : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = 2xe^{\alpha(x)}, \qquad x > 0;$$

(iii) B_f is the harmonic mean in I,

$$B_f(x,y)=\frac{2xy}{x+y}, \qquad x,y>0.$$

Beta-type functions as k-variable means

Theorem 3.

Let $k \in \mathbb{N}, k \ge 2$, be fixed, let $f : (0, \infty) \to (0, \infty)$ and $B_{f,k} : (0, \infty)^k \to (0, \infty)$ defined by

$$B_{f,k}(x_1,...,x_k) := rac{f(x_1)\cdots f(x_k)}{f(x_1+\cdots+x_k)}, \qquad x_1,...,x_k > 0.$$

The following conditions are equivalent: (i) $B_{f,k}$ is a mean in $(0,\infty)$; (ii) there is an additive function $\alpha : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = k^{\frac{1}{(k-1)^2}} \sqrt[k-1]{x} e^{\alpha(x)}, \quad x > 0;$$

(iii) $B_{f,k}$ is the beta-type mean, i.e.

$$B_{f,k}\left(x_{1},\ldots x_{k}\right)=\sqrt[k-1]{\frac{kx_{1}\cdots x_{k}}{x_{1}+\ldots+x_{k}}}, \quad x_{1},\cdots,x_{k}>0.$$

Martin Himmel (University of Zielona Góra) Cauchy quotient means and their properties

Definition 4.

For any $k \in \mathbb{N}$, $k \ge 2$, the function $\mathfrak{B}_k : (0,\infty)^k \to (0,\infty)$ defined by

$$\mathfrak{B}_k(x_1,\ldots,x_k) = \sqrt[k-1]{\frac{kx_1\cdots x_k}{x_1+\ldots+x_k}}, \quad x_1,\cdots,x_k > 0$$

is called the k-variable beta-type mean.

< Ξ > < Ξ >

The four classes of Cauchy quotients.

Cauchy quotients

beta-type function (exponential Cauchy quotient)

$$B_{f,k}(x_1,\ldots,x_k)=\frac{f(x_1)\cdot\ldots\cdot f(x_k)}{f(x_1+\ldots+x_k)}$$

logarithmic Cauchy quotient

$$L_{f,k}(x_1,\ldots,x_k) = \frac{f(x_1) + \ldots + f(x_k)}{f(x_1 \cdot \ldots \cdot x_k)}$$

• multiplicative (or power) Cauchy quotient

$$P_{f,k}(x_1,\ldots,x_k)=\frac{f(x_1)\cdot\ldots\cdot f(x_k)}{f(x_1\cdot\ldots\cdot x_k)}$$

additive Cauchy quotient

$$A_{f,k}(x_1,\ldots,x_k)=\frac{f(x_1)+\ldots+f(x_k)}{f(x_1+\ldots+x_k)}$$

where $f: I \to (0, \infty)$ is an arbitrary function defined on a suitable interval, and we asked:

- When is beta-type function $B_{f,k}$ a mean?
- When is a logarithmic Cauchy quotient $L_{f,k}$ a mean?
- When is a power Cauchy quotient $P_{f,k}$ a mean?
- When is an additive Cauchy quotient $A_{f,k}$ a mean?

> < 3> < 3> < 3> 3

where $f: I \to (0, \infty)$ is an arbitrary function defined on a suitable interval, and we asked:

- When is beta-type function $B_{f,k}$ a mean?
- When is a logarithmic Cauchy quotient $L_{f,k}$ a mean?
- When is a power Cauchy quotient $P_{f,k}$ a mean?
- When is an additive Cauchy quotient $A_{f,k}$ a mean? Answer:
- In each of the first three cases there exists *exactly one mean that can* be written in the form of a beta-type function, a logarithmic Cauchy quotient or a power Cauchy quotient, respectively.
- No mean of the form $A_{f,k}$ in any interval I.

個人 《문》 《문》 못날

When $L_{f,k}$ is a mean?

Theorem 5.

Let $k \in \mathbb{N}$, $k \ge 2$, be fixed, $f : (1, \infty) \to (0, \infty)$ be an arbitrary function. The following conditions are equivalent:

(i) the function $L_{f,k}:(1,\infty)^k\to (0,\infty)$ defined by

$$L_{f,k}(x_1,\ldots,x_k) := \frac{\sum\limits_{j=1}^k f(x_j)}{f\left(\prod\limits_{j=1}^k x_j\right)}, \qquad x_1,\ldots,x_k \in (1,\infty)$$

is a mean;

(ii) there is c > 0 such that

$$f(x) = \frac{c}{x^{\frac{1}{k-1}}} \log x, \qquad x \in (1,\infty);$$

When $L_{f,k}$ is a mean? (2)

Theorem 7 (continuation)

(iii) $L_{f,k}$ is of the form

$$L_{f,k}(x_1,...,x_k) = \frac{\sum_{i=1}^k \sqrt[k]{\prod_{j=1,j\neq i}^k x_j \log x_i}}{\sum_{i=1}^k \log x_i}, \qquad x_1,...,x_k \in (1,\infty).$$

When $L_{f,k}$ is a mean? (2)

Theorem 7 (continuation)

(iii) $L_{f,k}$ is of the form

$$L_{f,k}(x_1,...,x_k) = \frac{\sum_{i=1}^{k} \sqrt[k-1]{\prod_{j=1,j\neq i}^{k} x_j \log x_i}}{\sum_{i=1}^{k} \log x_i}, \qquad x_1,...,x_k \in (1,\infty).$$

Remark

An analogous result for $L_{f,k}$ holds true on the domain (0,1).

 The above mean for k = 2 belongs to the class of Beckenbach-Gini means.

(本國) (本臣) (王) (王) (王)

Theorem 6.

Let $k \in \mathbb{N}$, $k \ge 2$, be fixed and $f : (1, \infty) \to (0, \infty)$ continuous. The following conditions are equivalent:

(i) $P_{f,k}:(1,\infty)^k \to (0,\infty)$ defined by

$$P_{f,k}\left(x_1,\ldots,x_k\right) = \frac{f\left(x_1\right)\cdots f\left(x_k\right)}{f\left(x_1\cdots x_k\right)}, \quad x_1,\ldots,x_k > 1.$$

is a translative mean;

(ii) there exists $b \in \mathbb{R}$ such that

$$f(x) = x^{-\frac{\log\log x+b}{k\log k}}, \qquad x > 1;$$

Theorem 8 (continuation)

(iii) $P_{f,k}$ is of the form

$$P_{f,k}(x_1,\ldots,x_k) = \left(\prod_{j=1}^k x_j^{\log \frac{\log(x_1\cdot\ldots\cdot x_k)}{\log x_j}}\right)^{\frac{1}{k\log k}}, \quad x_1,\ldots,x_k > 1.$$

<□> <同> <同> <同> <同> <同> <同> の

Theorem 7.

Let $k \in \mathbb{N}$, $k \ge 2$, and a > 0 be fixed. There is no $f : [a, \infty) \to (0, \infty)$ such that the function $A_{f,k} : [a, \infty)^k \to (0, \infty)$ defined by

$$A_{f,k}(x_1,\ldots,x_k) := \frac{\sum\limits_{j=1}^k f(x_j)}{f\left(\sum\limits_{j=1}^k x_j\right)}, \qquad x_1,\ldots,x_k \ge a_j$$

or $\frac{1}{A_{f,k}}$ is a mean.

個人 시민지 시민지 못 먹

- Beta-type functions naturally generalize the Euler Beta function.
- A two-variable beta-type function is a mean if, and only if, it is the harmonic mean.
- Beta-type functions of k-variables give a homogeneous mean, called beta-type mean, which is neither harmonic nor quasi-arithmetic for k ≥ 3.
- $L_{f,k}$ and $\frac{1}{L_{f,k}}$ exhibit means related to Beckenbach-Gini means.
- There exists a mean in terms of $P_{f,k}$ and $\frac{1}{P_{f,k}}$.
- There does not exist any mean of the form $A_{f,k}$ or $\frac{1}{A_{f,k}}$.

< □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □

Remark 1

Let $k \in \mathbb{N}$, $k \ge 2$; $a \ge 0$, and I be as in the Definition 1, and let $f, g: I \to (0, \infty)$. The beta-type functions have the following properties: (i) (equality) $B_{f,k} = B_{g,k}$ iff there is a function $E: \mathbb{R} \to (0, \infty)$ such that $\frac{g}{f} = E|_{I}$ and E is exponential, i.e.

$$\operatorname{E}\left(x+y
ight)=\operatorname{E}\left(x
ight)\operatorname{E}\left(y
ight)$$
 , $x,y\in\mathbb{R};$

(ii) (multiplicativity) for all $f,g:(a,\infty) \to (0,\infty)$,

$$B_{f \cdot g,k} = B_{f,k} \cdot B_{g,k};$$

(iii) for every $f:(a,\infty)
ightarrow (0,\infty)$,

$$B_{\frac{1}{f},k}=\frac{1}{B_{f,k}}.$$

Question

What are properties of the *k*-variable beta-type mean?

Remark 2

Let $k \in \mathbb{N}$, $k \ge 2$ be fixed. The beta-type mean has the following properties:

(i) \mathfrak{B}_k is homogeneous, i.e.

$$\mathfrak{B}_k(tx_1,\ldots,tx_k) = t\mathfrak{B}_k(x_1,\ldots,x_k), \qquad x_1,\ldots,x_k, t > 0.$$

(ii) \mathfrak{B}_k is quasi-arithmetic, i.e. there is a continuous and strictly monotone function $\varphi : (0, \infty) \to \mathbb{R}$ such that

$$\mathfrak{B}_{k}\left(x_{1},\ldots x_{k}
ight)=arphi^{-1}\left(rac{arphi\left(x_{1}
ight)+\ldots+arphi\left(x_{k}
ight)}{k}
ight), \qquad x_{1},\ldots,x_{k}>0,$$

if, and only if, k = 2. Moreover, for k = 2, this this equality holds true iff $\varphi(t) = \frac{a}{t} + b$ for some real $a, b, a \neq 0$, and \mathfrak{B}_2 is the harmonic mean:

$$\mathfrak{B}_2(x,y)=\frac{2xy}{x+y}, \qquad x,y>0.$$

3 5

< ロ > < 同 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ >

A characterization of \mathfrak{B}_k in the class of premeans of beta-type

Using the Krull result on difference equations, employing some convexity condition on f, it is possible to obtain another characterization of beta-type premeans.

Theorem 8.

Let $k \in \mathbb{N}$, $k \ge 2$; and $a \ge 0$ be fixed, and let $I = (a, \infty)$, if $a \ge 0$; or $I = [a, \infty)$, if a > 0. Assume that $f : I \to (0, \infty)$ is differentiable and such that the function $\frac{f'}{f} \circ \exp$ is convex. Then the following conditions are equivalent

(i) the beta-type function $B_{f,k}$ is a premean in I;

(ii) there is $c \in \mathbb{R}$ such that

$$f(x) = k^{\frac{1}{(k-1)^2}} \sqrt[k-1]{x} e^{cx}, \qquad x \in I;$$

(iii) $B_{f,k} = \mathfrak{B}_k$.

Theorem 9.

Let a $\geq -\infty$ be arbitrarily fixed. Suppose that $F:(a,\infty)\to \mathbb{R}$ is convex or concave, and

$$\lim_{x\to\infty} \left[F\left(x+1\right)-F\left(x\right)\right]=0.$$

Then for every fixed $(x_{0,}, y_0) \in (a, \infty) \times \mathbb{R}$ there exists exactly one convex or concave function $\varphi : (a, \infty) \to \mathbb{R}$ satisfying the functional equation

$$\varphi(x+1) = \varphi(x) + F(x), \qquad x > a$$
 (4)

such that

$$\varphi(x_0)=y_0;$$

Theorem 5

moreover, for all x > a,

$$\varphi(x) = y_0 + (x - x_0) F(x_0)$$
 (5)

$$-\sum_{n=0}^{\infty} \left\{ F(x+n) - F(x_0+n) - (x-x_0) \left[F(x_0+n+1) - F(x_0+n) \right] \right\}.$$

< □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □

A second characterization of \mathfrak{B}_k

Applying the theory of iterative functional equations for functions of the class C^n , we obtain another characterization of the *k*-variable beta-type mean.

Theorem 10.

Let $k \in \mathbb{N}$, $k \ge 2$ be fixed. Assume that $f : (0, \infty) \to (0, \infty)$ is of the class C^2 and the function

$$(0,\infty) \ni x \longmapsto \frac{f(x)}{x^{\frac{1}{k-1}}}$$

has an extension to a class C^2 in the interval $[0, \infty)$. Then the following conditions are equivalent (i) the beta-type function $B_{f,k}$ is a premean in $(0, \infty)$; (ii) there is $c \in \mathbb{R}$ such that

$$f(x) = k^{\frac{1}{(k-1)^2}} \sqrt[k-1]{x} e^{cx}, \qquad x > 0;$$

In the this result we determine the functions which are affine with respect to the mean \mathfrak{B}_k for every $k \in \mathbb{N}$, $k \ge 2$.

Theorem 11.

A function $f : (0, \infty) \to (0, \infty)$ is affine with respect to the family of means $\{\mathcal{B}_k : k \in \mathbb{N}, k \geq 2\}$, i.e.

$$f\left(\mathcal{B}_{k}\left(x_{1},...,x_{k}
ight)
ight)=\mathcal{B}_{k}\left(f\left(x_{1}
ight),...,f\left(x_{1}
ight)
ight), \qquad x_{1},...,x_{k}>0; \ k\in\mathbb{N},\ k\geq2,$$

iff either f is linear, i.e. f(x) = f(1)x for all x > 0, or f is constant.

The proof relies on the fact that $\mathcal{B}_2 = H$ is the harmonic mean, which is quasi-arithmetic. The affine functions of quasi-arithmetic means are easy to determine. The problem to find all functions $f : (0, \infty) \to (0, \infty)$ which are affine with respect to the beta-type mean \mathcal{B}_k for a fixed $k \in \mathbb{N}$, $k \ge 3$, remains open.

◆□ ▶ ▲ □ ▶ ▲ □ ▶ ▲ □ ▶ ● ● ●

Remark 3

Let $I \subset \mathbb{R}$ be an interval and $\varphi : I \to \mathbb{R}$ be one-to-one and onto. A function $f : I \to \mathbb{R}$ satisfies equation

$$f\left(arphi^{-1}\left(rac{arphi\left(x
ight) +arphi\left(y
ight) }{2}
ight)
ight) =arphi^{-1}\left(rac{arphi\left(f\left(x
ight)
ight) +arphi\left(f\left(y
ight)
ight) }{2}
ight) ,\qquad x,y\in I,$$

if, and only if, there exist an additive function $\alpha:\mathbb{R}\to\mathbb{R}~$ and $b\in\mathbb{R}$ such that

$$f(x) = \varphi^{-1}(\alpha(\varphi(x)) + b), \qquad x \in I.$$

Remark 4

A function $f: (0,\infty) \to (0,\infty)$ is affine with respect to the mean \mathfrak{B}_2 , i.e.

$$f\left(\mathfrak{B}_{2}\left(x,y\right)\right)=\mathfrak{B}_{2}\left(f\left(x
ight),f\left(y
ight)
ight),\qquad x,y>0,$$

if, and only if, there exist $p, q \ge 0, p + q > 0$, such that

$$f(x) = \frac{x}{p+qx}, \qquad x > 0.$$

.

Definition 12.

Let X be a real linear space, $C \subset X$ a convex set and $k \in \mathbb{N}$. A function $M : C^k \to X$ is called k-variable mean in C if there exist functions $\lambda_i : C^k \to [0, +\infty)$, for i = 1, ..., k, such that, for all $\mathbf{x}_1, ..., \mathbf{x}_k \in C$,

$$M(\mathbf{x}_1,\ldots,\mathbf{x}_k)=\sum_{i=1}^k\lambda_i(\mathbf{x}_1,\ldots,\mathbf{x}_k)\mathbf{x}_i,$$

and

$$\sum_{i=1}^k \lambda_i \left(\mathbf{x}_1, \ldots, \mathbf{x}_k \right) = 1.$$

<撮▶ < ≧▶ < ≧▶ . 黒目 の

Let $M, N: I^2 \to I$ be given means. A mean $K: I^2 \to I$ is called (M, N)-invariant, if

$$K(M(x,y),N(x,y)) = K(x,y), \qquad x,y \in I.$$

레이 소문이 소문이 모님

Thank You for your attention



Martin Himmel (University of Zielona Góra) Cauchy quotient means and their properties

- M. Kuczma, Functional equations in a single variable, Monografie Matematyczne V. 46, Polish Scientific Publishers, Warszawa, 1968.
- M. Kuczma, B. Choczewski, R. Ger, Iterative Functional Equations, Cambridge University Press, Cambridge, (1990).
- M. Himmel, J. Matkowski, Homogeneous beta-type functions (J. Class. Anal., Volume 10, Number 1 (2017), 59–66.).
- M. Himmel, J. Matkowski, Beta-type means (to appear in J. Difference Equ. Appl.)
- Young Whan Lee, Approximate pexiderized gamma-beta type functions, Journal of Inequalities and Applications 2013, 2013:14.