Cantor bouquets in spiders' webs

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Basic definitions

The **Fatou set**, $F(f)$, is the set of points for which there is a neighbourhood where the family of iterates is equicontinuous.

The **Julia set**, $J(f)$, is the complement of the Fatou set.

The **escaping set**, $I(f)$, is the set of points that tend to infinity under iteration.

The **fast escaping set**, $A(f)$, consists of the points that escape to infinity 'as fast as possible':

$$A(f) = \cup_{n \in \mathbb{N}} f^{-n}(A_R(f)),$$

where, for $R > 0$ sufficiently large,

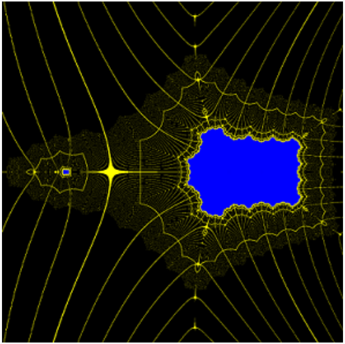
$$A_R(f) = \{z : |f^n(z)| \geq M^n(R) \text{ for all } n \in \mathbb{N}\}.$$

Cantor bouquets and spiders' webs

Part of the escaping set of
$$z \mapsto \frac{1}{4}e^z.$$

A Cantor bouquet.

Part of the escaping set of
$$z \mapsto \frac{1}{2}(\cos z^{1/4} + \cosh z^{1/4}).$$



A spider's web.

Straight brushes

Definition

A subset B of $[0, +\infty) \times (\mathbb{R} \setminus \mathbb{Q})$ is called a *straight brush* if the following properties are satisfied:

- The set B is a closed subset of $\mathbb{R}^2$.
- For every $(x, y) \in B$ there exists $t_y \geq 0$ such that $\{x : (x, y) \in B\} = [t_y, +\infty)$. The set $[t_y, +\infty) \times \{y\}$ is called the hair attached at y and the point (t_y, y) is called the endpoint.
- The set $\{y : (x, y) \in B \text{ for some } x\}$ is dense in $\mathbb{R} \setminus \mathbb{Q}$. Moreover, for every $(x, y) \in B$ there exist two sequences of hairs attached respectively at $\beta_n, \gamma_n \in \mathbb{R} \setminus \mathbb{Q}$ such that $\beta_n < y < \gamma_n$, $\beta_n, \gamma_n \to y$ and $t_{\beta_n}, t_{\gamma_n} \to t_y$ as $n \to \infty$.

Cantor bouquets

Definition

A *Cantor bouquet* is any set ambiently homeomorphic to a straight brush.

Examples of functions that admit Cantor bouquets in their Julia sets:

- λe^z, $0 < \lambda < 1/e$;
- $\mu \sin z$, $0 < \mu < 1$;
- certain functions with a bounded set of critical and asymptotic values (i.e. in the Eremenko-Lyubich class).

Spiders' webs

Definition

A set $E \subset \mathbb{C}$ is called a *spider's web* if it is connected and there exists a sequence of bounded simply connected domains G_n with $G_n \subset G_{n+1}$ for $n \in \mathbb{N}$, $\partial G_n \subset E$ for $n \in \mathbb{N}$, and $\cup_{n \in \mathbb{N}} G_n = \mathbb{C}$.

Examples of functions whose escaping and fast escaping sets are spiders' webs:

- functions with multiply connected Fatou components;
- functions of small growth;
- functions defined by certain gap series; and
- many functions exhibiting the pits effect.

A Cantor bouquet in a spider's web

Let $f : \mathbb{C} \to \mathbb{C}$ with $f(z) = \cos z + \cosh z$. It is known that $I(f)$ is a spider's web. We will prove that there exists a Cantor bouquet in $I(f)$. In fact, it is a subset of $J(f)$ and $A(f)$ as well.

The idea is to study points in $I(f)$ that remain in certain strips under iteration and take advantage of the detailed dynamics of f to locate an uncountable number of pairwise disjoint curves inside said strips.

A Cantor bouquet in a spider's web

The method used to locate a Cantor bouquet is as follows.

- For fixed $N \in \mathbb{N}$, define $2N+1$ horizontal half-strips of width $\pi/2$ in the right half-plane; $\{T_k : k = -N, \ldots, N\}$.
- Let Λ_N be the points that stay in $\cup_{k=-N}^{N} T_k$ under iteration. The sequence of integers $s_0 s_1 \ldots$ defined by

  $$f^n(z) \in T_{s_n}$$

  is called the *itinerary* of z.
- We prove that to each sequence of integers with absolute values less than or equal to N, there corresponds a unique curve in Λ_N with the property that each point in this curve has this same sequence as an itinerary.

A Cantor bouquet in a spider's web

(cont.)

- This gives a one-to-one correspondence between the set of curves and a Cantor set. The closure of the union of the curves found for each $N \in \mathbb{N}$ is a Cantor bouquet.

This is the technique used to accomplish the above:

- Consider a rectangle of length 2π that lies in some T_k and map it forward under f, finding a further number of similar rectangles in its image.
- This allows us to make a choice of one rectangle, which corresponds to one integer.
- We then iterate this process.

A Cantor bouquet in a spider's web

The Cantor bouquet we have found is contained in $I(f)$. It is a simple task to show that it is, in fact, a subset of $A(f)$ as well.

Finally, the Cantor bouquet is also contained in $J(f)$. This is slightly trickier to prove and uses a result on the expansion property of f', as well as a distortion lemma for open sets in Fatou components.

Plans for future work

- Extend the results to the families of transcendental entire functions defined, for $n \geq 3$, by

$$\mathcal{E}_n = \left\{ f : f(z) = \sum_{k=0}^{n-1} a_k \exp\left(\omega_n^k z\right) \right\},$$

  where $a_k \neq 0$ for $k \in \{0, 1, \ldots, n-1\}$ and $\omega_n = \exp(2\pi i/n)$ is an nth root of unity.

- Broaden the study to other areas of symbolic dynamics, e.g. coding trees of preimages.