Entire functions arising from trees

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Mathematisches Seminar, CAU Kiel

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Outline

An inverse problem

Shabat entire functions

Definition

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- For any Shabat entire function f, put $T_f := f^{-1}([-1, 1])$.

Trees arising from entire functions

Observation

Let f be a Shabat entire function. Then T_f is a tree in the plane.

(1) *If f is a polynomial, then T_f is a finite tree;*

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- Examples: $z \mapsto 4z^3 - 3z$; $z \mapsto \sin(z)$.
- T_f is called a true tree if f is a Shabat entire function.
- Two trees T_1 and T_2 in the plane (not necessarily being true trees) are equivalent, if there is a homeomorphism $\varphi : \mathbb{C} \to \mathbb{C}$ such that $\varphi(T_1) = T_2$.

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- If there is such a true tree, then we call T_f a *true form* of T.

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"Theorem" (Nevanlinna)

Any homogeneous tree of valence ≥ 3 does not have a true form.

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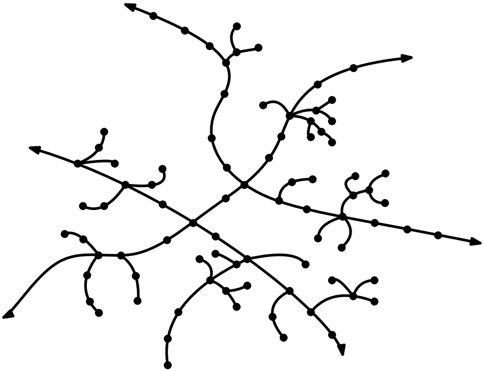

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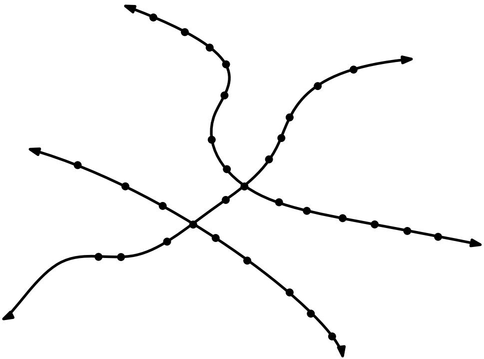

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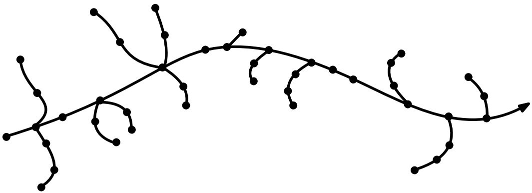

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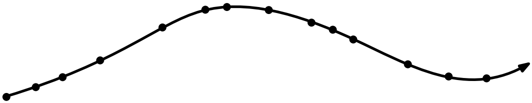

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Remark

A connected, infinite and locally finite graph Γ, endowed with the word metric, is a geodesic metric space.

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Theorem (Cui, 2017)

Any tree satisfying the topological uniformness condition has a true form.

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- *(Sharpness) Every item in the topological uniformness condition cannot be dropped.*
- *(Extension) Every item can be generalized.*

Type problem

Überlagerungsfläche

Theorem (Conformal uniformization)

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Stoïlow: There is a unique conformal structure on X which makes X a *Riemann surface*.

Meromorphic functions

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- If X is parabolic, then there is a conformal map $\phi : \mathbb{C} \to X$ such that $f := p \circ \phi : \mathbb{C} \to \widehat{\mathbb{C}}$ is meromorphic.
- If X is hyperbolic, then there is a conformal map $\psi : \mathbb{D} \to X$ such that $g := p \circ \psi : \mathbb{D} \to \widehat{\mathbb{C}}$ is meromorphic.

Surfaces of class $\mathcal{S}$

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Definition

A surface (X, p) spread over the sphere belongs to class $\mathcal{S}$, if there are $q < \infty$ points $A := \{a_1, \ldots, a_q\}$ such that

$$p : X \setminus p^{-1}(A) \to \widehat{\mathbb{C}} \setminus A$$

is a covering map.

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- Conversely, Speiser graphs provide a combinatorial pattern to construct surfaces spread over the sphere in class $\mathcal{S}$.

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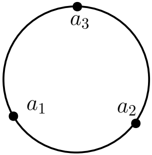

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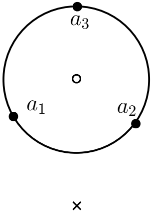

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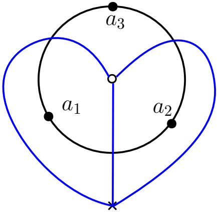

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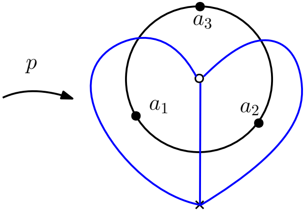

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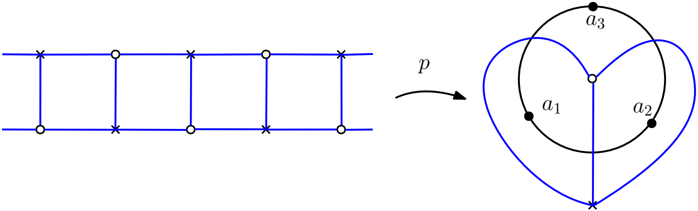

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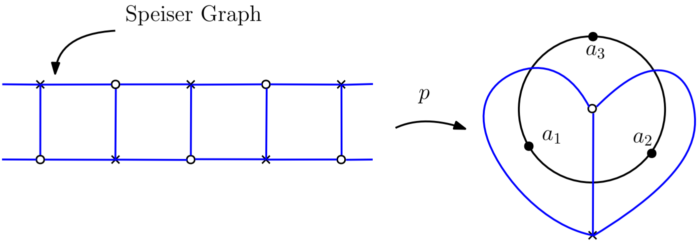

Realization of entire functions

Quasi-isometry

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Let (X_1, d_1) and (X_2, d_2) be two metric spaces. A map $\Phi : X_1 \to X_2$ is called a quasi-isometry, if it satisfies the following two conditions:

(1) *for some $\varepsilon > 0$, the ε-neighborhood of the image of Φ in X_2 covers X_2;*

(2) *there are constants $k \geq 1$, $C \geq 0$ such that for all $x_1, x_2 \in X_1$,*

$$\frac{1}{k} \cdot d_1(x_1, x_2) - C \leq d_2(\Phi(x_1), \Phi(x_2)) \leq k \cdot d_1(x_1, x_2) + C.$$

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Example

The two dimensional lattice $\mathbb{Z} \times \mathbb{Z}$ is quasi-isometric to $\mathbb{E}^2$.

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Proposition

Two connected finite valence graph which are quasi-isometric are simultaneously hyperbolic or parabolic.

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Theorem (DM criterion)

Let $n \in \mathbb{N}$ be fixed. A surface spread over the sphere $(X, p) \in \mathcal{S}$ is parabolic if and only if Γ_n is parabolic.

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$$T \xrightarrow{\text{Triangulation}}$$

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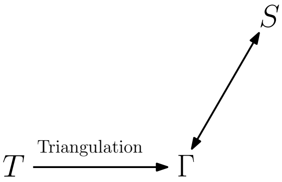

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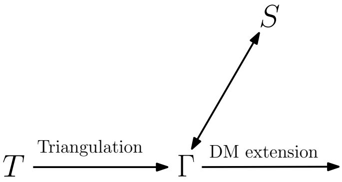

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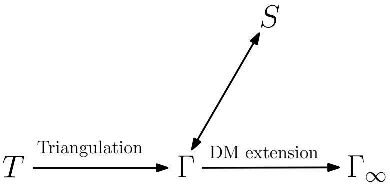

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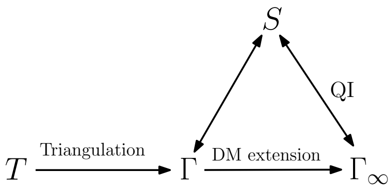

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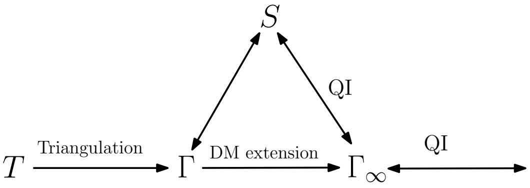

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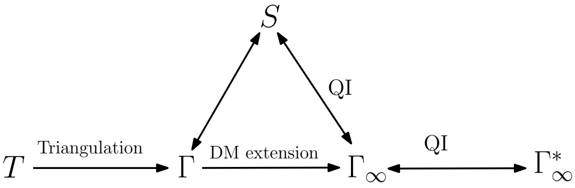

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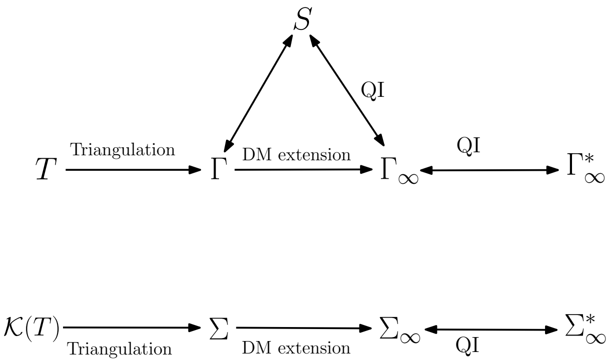

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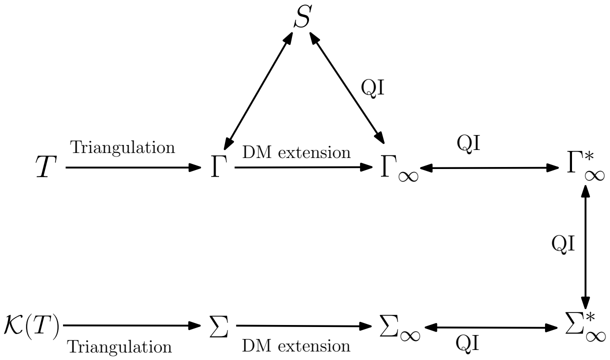

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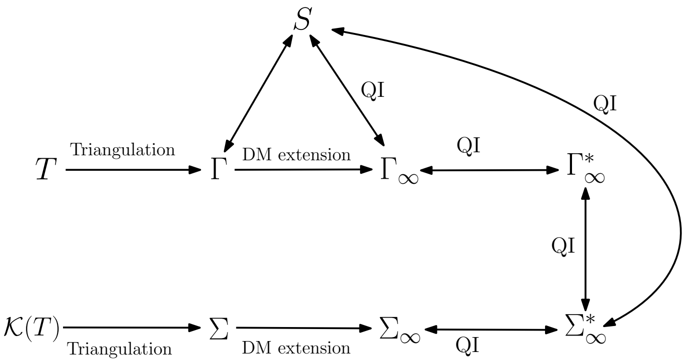

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