Hairs of a higher-dimensional analogue of the exponential family

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Outline

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1. Hairs of entire functions
2. Quasiregular maps

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For an attracting fixed point $\xi \in \mathbb{C}$ of an entire function f

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For $0 < \lambda < 1/e$ the function $E_\lambda(z)$ has an attracting fixed point $\xi_\lambda \in \mathbb{R}$.

Theorem (Devaney, Krych 1984)

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For $0 < \lambda < 1/e$ we have $\mathcal{J}(E_\lambda) = \mathbb{C} \setminus A(\xi_\lambda)$ and $\mathcal{J}(E_\lambda)$ is a "Cantor set of curves".

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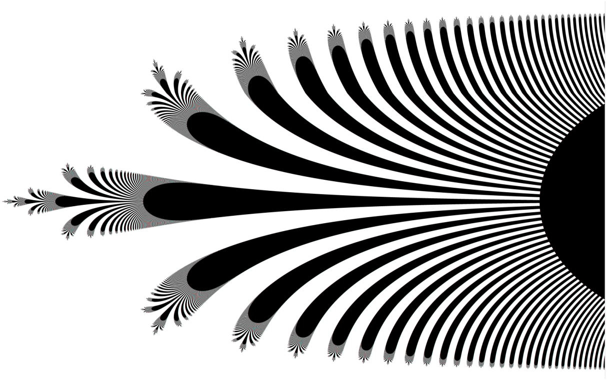

Figure: Part of $\mathcal{J}(E_\lambda)$ for $\lambda = 1/4$.

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- Show that the equivalence classes are hairs.

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- Barański (2007): For a disjoint type map f of finite order, the set $\mathcal{J}(f)$ is a Cantor Bouquet.
- Rottenfußer, Rückert, Rempe, Schleicher (2011): For a function f of bounded type and of finite order, the set $\mathcal{J}(f)$ contains an uncountable union of hairs.

Quasiregular maps

Motivation for quasiregular maps

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Let $G \subset \mathbb{R}^d$ be a domain and let $f \in W^{1,d}_{loc}(G)$ be continuous. We say that f is **quasiregular** if there exists a constant $K := K(f) \geq 1$ such that

$$\|Df(x)\|^d \leq K J_f(x) \quad \text{a.e.}$$

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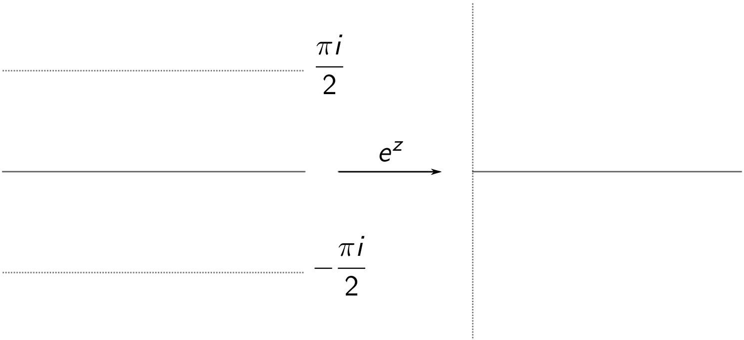

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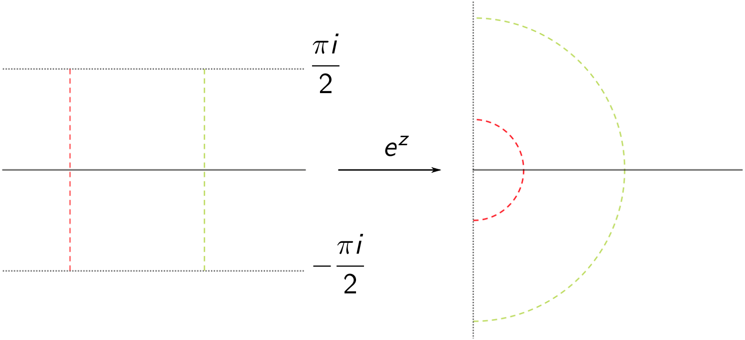

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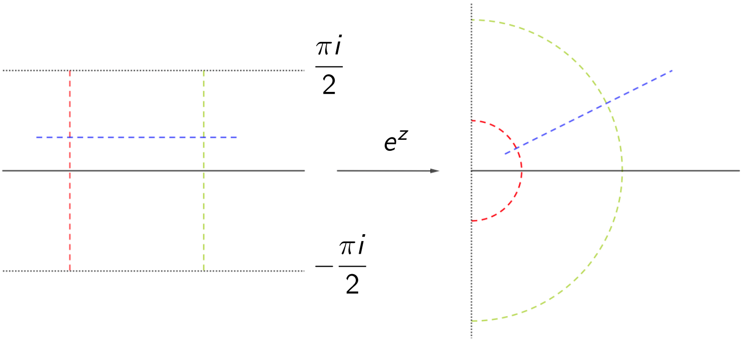

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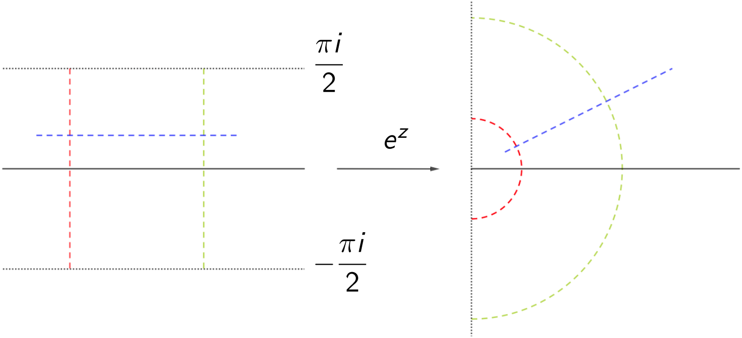

With

$$h\colon [-\pi/2, \pi/2] \to \mathbb{C},\ h(y) = \cos y + i \sin y$$

and $z = x + iy$ we obtain

$$e^z = e^x h(y).$$

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Following Iwaniec and Martin, we use a bi-Lipschitz map $h\colon Q \to S_+$, where $Q := [-1,1]^{d-1}$ and

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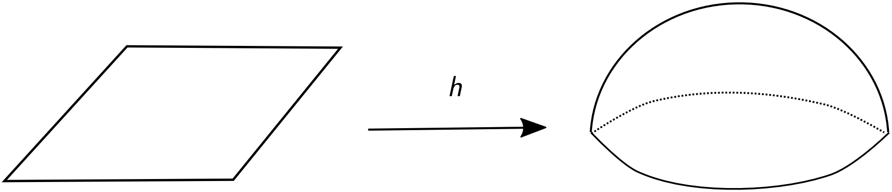

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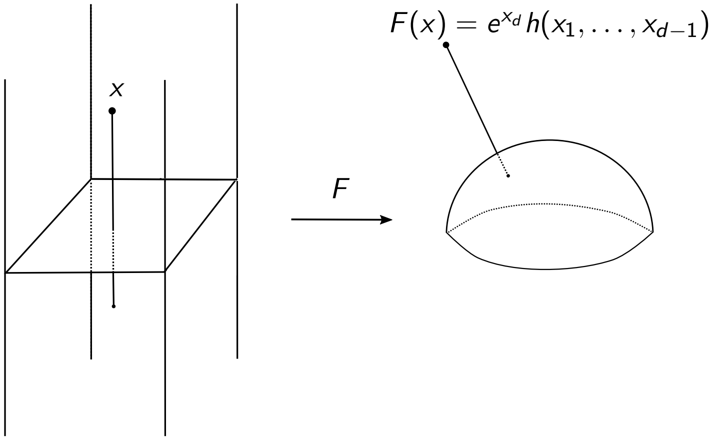

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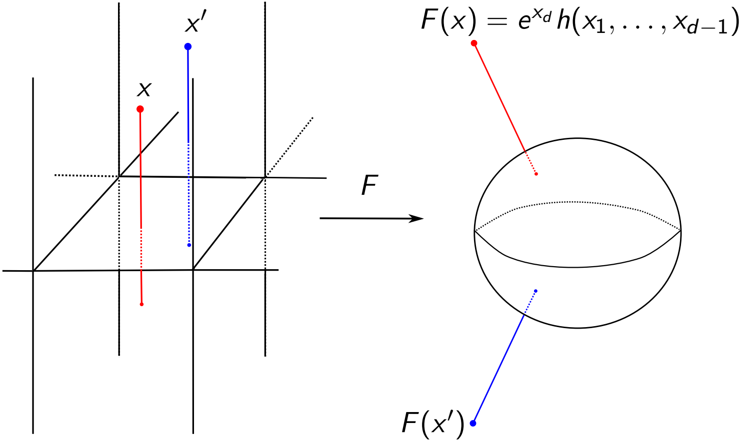

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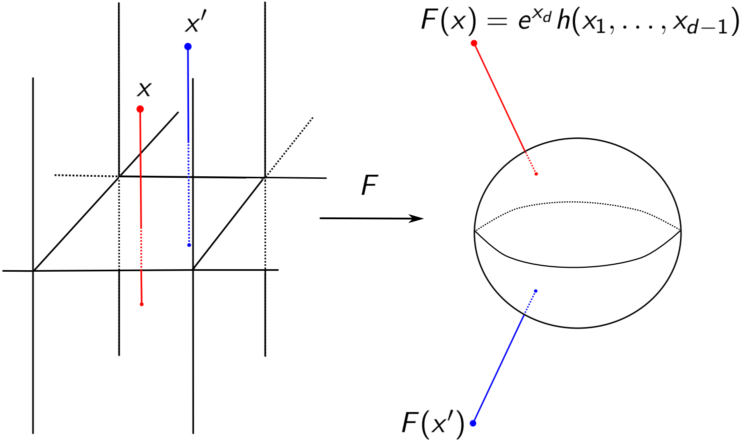

Via reflections we can extend F to $\mathbb{R}^d$ which we call a **Zorich map**.

Zorich maps with one attracting fixed point

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Let f be as above. Then there exists a unique fixed point $\xi = (\xi_1, \ldots, \xi_d)$ and the set

$$J := \{x \in \mathbb{R}^d \colon f^k(x) \nrightarrow \xi\}$$

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Idea of the proof: Obtain the hairs as a locally uniform limit of a sequence of certain curves

Differentiability of hairs

Differentiability of hairs for exponential maps

[1]Reference: L. Rempe, Dynamics of exponential maps, doctoral thesis, Christian-Albrechts-Universität Kiel (2003), p. 34

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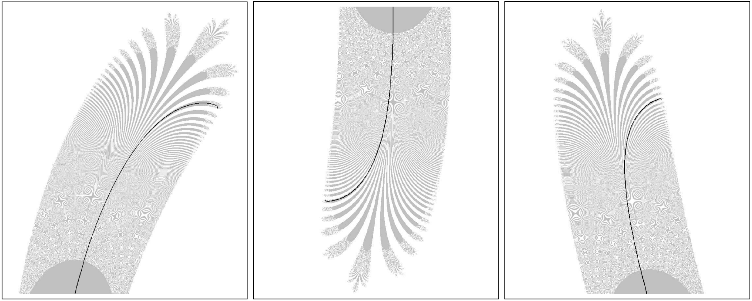

Figure: Example of a nondifferentiable endpoint for $f(z) = e^z - 2$.[1]

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Theorem 2 (C. 2017)

There exists a function of bounded type and of finite order, where every hair is nowhere differentiable.

Thank you for your attention!