Stability of the Denjoy–Wolff theorem

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Preliminaries

$\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$

$\mathcal{H}(\mathbb{D}) = \{g \colon \mathbb{D} \to \mathbb{D} \text{ holomorphic}\}$

Endow $\mathcal{H}(\mathbb{D})$ with the topology of locally uniform convergence.

Iteration

For a function $f \in \mathcal{H}(\mathbb{D})$, the n^{th} *iterate of f* is the function

$$f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}.$$

Theorem (Denjoy–Wolff)

Let $f \in \mathcal{H}(\mathbb{D})$. Assume that f is not an elliptic Möbius map, then there exists $p \in \overline{\mathbb{D}}$ such that $f^n \to p$ locally uniformly on $\mathbb{D}$.

The point p is called *the Denjoy–Wolff point of f*.

Elliptic Möbius maps

Let $f(z) = e^{i\pi\theta} z$, where $\theta \in \mathbb{R}$.

If $\theta \in \mathbb{Q}$, then the set $\{f^n(z_0) \colon n \in \mathbb{N}\}$ is finite.

If $\theta \in \mathbb{R} \setminus \mathbb{Q}$, then the set $\{f^n(z_0) \colon n \in \mathbb{N}\}$ is dense in the circle of radius $|z_0|$ centred at 0.

Composition sequences

Let $\{f_n\}$ be a sequence in $\mathcal{H}(\mathbb{D})$.

The *composition sequence generated by* $\{f_n\}$ is the sequence

$$F_n = f_1 \circ f_2 \circ \cdots \circ f_n.$$

Examples, for $n = 2, 3, \cdots$

$$f_n(z) = \left(1 - \frac{1}{n^2}\right) z, \quad F_n(z) = z \prod_{k=2}^{n} \left(1 - \frac{1}{k^2}\right) \longrightarrow \tfrac{1}{2} z$$

$$g_n(z) = \left(1 - \frac{1}{n}\right) z, \qquad G_n(z) = z \prod_{k=2}^{n} \left(1 - \frac{1}{k}\right) \longrightarrow 0$$

Composition sequences

Theorem

Let $\{f_n\}$ be a sequence in $\mathcal{H}(\mathbb{D})$. Assume that there exists a compact set $K \subset \mathbb{D}$, such that $f_n(\mathbb{D}) \subset K$. Then $F_n = f_1 \circ \cdots \circ f_n$ converges locally uniformly to a constant in $\mathbb{D}$.

Problem

Let $\{f_n\}$ be a sequence in $\mathcal{H}(\mathbb{D})$, such that $f_n \to f \in \mathcal{H}(\mathbb{D})$ locally uniformly.

$f^n = f \circ \cdots \circ f \quad \rightsquigarrow$ Denjoy–Wolff theorem/

$F_n = f_1 \circ \cdots \circ f_n \quad \rightsquigarrow ?$

Cases:

f is an elliptic Möbius map

f has its Denjoy–Wolff point inside $\mathbb{D}$

f has its Denjoy–Wolff point on $\partial\mathbb{D}$

Elliptic case

Examples, for $n = 2, 3, \cdots$

$$f_n(z) = e^{i\pi\theta}\left(1 - \frac{1}{n^2}\right)z, \quad F_n(z) = e^{i(n-1)\pi\theta} z \prod_{k=2}^{n}\left(1 - \frac{1}{k^2}\right)$$

$$g_n(z) = e^{i\pi\theta}\left(1 - \frac{1}{n}\right)z, \qquad G_n(z) = e^{i(n-1)\pi\theta} z \prod_{k=2}^{n}\left(1 - \frac{1}{k}\right)$$

Elliptic case

Theorem

Let $\{f_n\}$ be a sequence in $\mathcal{H}(\mathbb{D})$. Assume that there exist $z_1, z_2 \in \mathbb{D}$ distinct such that

$$\sum_{n=1}^{\infty} \rho(f_n(z_i), e^{i\pi\theta} z_i) < \infty, \quad \text{for } i = 1, 2\,,$$

where $\theta \in \mathbb{Q}$. Then $F_n = f_1 \circ \cdots \circ f_n$ has finitely many limit functions.

Non-elliptic case

Let $\{f_n\}$ be a sequence in $\mathcal{H}(\mathbb{D})$, and let f be a function in $\mathcal{H}(\mathbb{D})$, which is not an elliptic Möbius map.

Denote by p the Denjoy–Wolff point of f.

Theorem

If f has $p \in \mathbb{D}$ as its Denjoy–Wolff point, then there exists a neighbourhood $\mathcal{N}$ of f in $\mathcal{H}(\mathbb{D})$ such that if $f_n \in \mathcal{N}$, then $F_n = f_1 \circ \cdots \circ f_n$ converges locally uniformly to a constant in $\mathbb{D}$.

Non-elliptic case

Theorem

Assume that f has $p \in \partial\mathbb{D}$ as its Denjoy–Wolff point. Then for every sequence of neighbourhoods $\{\mathcal{N}_m\}$ of f, there exists a sequence of functions $\{f_m\}$, such that $f_m \in \mathcal{N}_m$ and $F_m = f_1 \circ \cdots \circ f_m$ diverges.